Available online at www.sciencedirect.com

ScienceDirect

Expo. Math. 31 (2013) 350–367

EXPOSITIONES MATHEMATICAE

www.elsevier.com/locate/exmath

Formal Laurent series in several variables

Ainhoa Aparicio Monforte[a], Manuel Kauers[b,*]

[a] *Université Lille 1 Sciences et Technologies, 59655 Villeneuve d'Ascq, Cedex, France*
[b] *Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, A4040 Linz, Austria*

Received 5 August 2012

Abstract

We explain the construction of fields of formal infinite series in several variables, generalizing the classical notion of formal Laurent series in one variable. Our discussion addresses the field operations for these series (addition, multiplication, and division), the composition, and includes an implicit function theorem.

MSC 2010: primary 13F25; secondary 05A15

Keywords: Formal power series

1. Introduction

The purpose of this article is twofold. In the first part (Sections 2–4), we explain how to construct fields of formal Laurent series in several variables. This part has an expository flavour. The construction we present is not new; similar constructions can already be found in the literature. However, the justification of their validity is usually kept brief or more abstract than necessary. We have found it instructive to formulate the arguments in a somewhat more concrete and expanded way, and we include these proofs here in the hope that this may help to demystify and popularize the use of formal Laurent series in several variables. The results in the second part (Sections 5–6) seem to be new. We discuss there the circumstances under which we can reasonably define the composition of multivariate

* Corresponding author.
E-mail addresses: ainhoaaparicio@gmail.com (A. Aparicio Monforte), mkauers@risc.jku.at (M. Kauers).

0723-0869
http://dx.doi.org/10.1016/j.exmath.2013.01.004

formal Laurent series, and we present a version of the implicit function theorem applicable to multivariate formal Laurent series.

Recall the situation of a single variable. The set $\mathbb{K}[[x]]$ of formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in some field $\mathbb{K}$ forms an integral domain together with the usual addition and multiplication. Such a series $f(x)$ admits a multiplicative inverse $g(x) \in \mathbb{K}[[x]]$ if and only if $a_0 \neq 0$ (see, e.g., [14,10]). If $f(x)$ is any nonzero element of $\mathbb{K}[[x]]$, then not all its coefficients are zero, and if e is the smallest index such that $a_e \neq 0$, then we have $f(x) = x^e h(x)$ for some $h(x) \in \mathbb{K}[[x]]$ which admits a multiplicative inverse. The object $x^{-e}h(x)^{-1}$ qualifies as a multiplicative inverse of $f(x)$. In the case of a single variable, we may therefore define $\mathbb{K}((x))$ as the set of all objects $x^e h(x)$ where e is some integer and $h(x)$ is some element of $\mathbb{K}[[x]]$. Then $\mathbb{K}((x))$ together with the natural addition and multiplication forms a field. This is the field of formal Laurent series in the case of one variable.

The case of several variables is more subtle. The set $\mathbb{K}[[x, y]]$ of formal power series $f(x, y) = \sum_{n,k=0}^{\infty} a_{n,k} x^n y^k$ in two variables x, y with coefficients in $\mathbb{K}$ also forms an integral domain, and it remains true that an element $f(x, y) \in \mathbb{K}[[x, y]]$ admits a multiplicative inverse if and only if $a_{0,0} \neq 0$. But in general, it is no longer possible to write an arbitrary power series $f(x, y)$ in the form $f(x, y) = x^{e_1} y^{e_2} h(x, y)$ where $h(x, y) \in \mathbb{K}[[x, y]]$ admits a multiplicative inverse in $\mathbb{K}[[x, y]]$. As an example, consider the series $f(x, y) = x + y = x^1 y^0 + x^0 y^1 \in \mathbb{K}[[x, y]]$. If we want to write $f(x, y) = x^{e_1} y^{e_2} h(x, y)$ for some $h(x, y) \in \mathbb{K}[[x, y]]$, we have $h(x, y) = x^{1-e_1} y^{-e_2} + x^{-e_1} y^{1-e_2}$. In order for $h(x, y)$ to have a nonzero constant term, we can only choose $(e_1, e_2) = (1, 0)$ or $(e_1, e_2) = (0, 1)$, but for these two choices, $h(x, y)$ is $1 + x^{-1}y$ or $xy^{-1} + 1$, respectively, and none of them belongs to $\mathbb{K}[[x, y]]$.

There are at least three possibilities to resolve this situation. The first and most direct way is to consider fields of iterated Laurent series [17, Chapter 2], for instance the field $\mathbb{K}((x))((y))$ of univariate Laurent series in y whose coefficients are univariate Laurent series in x. Clearly this field contains $\mathbb{K}[[x, y]]$, and the multiplicative inverse of $x + y$ in $\mathbb{K}((x))((y))$ is easily found via the geometric series to be

$$\frac{1}{x+y} = \frac{1/x}{1-(-y/x)} = \sum_{n=0}^{\infty} (-1)^n x^{-n-1} y^n.$$

Of course, viewing $x + y$ as an element of $\mathbb{K}((y))((x))$ leads to a different expansion.

The second possibility is more abstract. This construction goes back to Malcev [11] and Neumann [13] (see [15,17] for a more recent discussion). Start with an abelian group G (e.g., the set of all power products $x_1^{i_1} \cdots x_p^{i_p}$ with exponents $i_1, \ldots, i_p \in \mathbb{Z}$ and the usual multiplication) and impose on the elements of G some order $\preccurlyeq$ which respects multiplication (see Section 3 below for definitions and basic facts). Define $\mathbb{K}((G))$ as the set of all formal sums

$$a = \sum_{g \in G} a_g g$$

with $a_g \in \mathbb{K}$ for all $g \in G$ and the condition that their supports $\operatorname{supp}(a) := \{ g \in G \mid a_g \neq 0 \}$ contain no infinite strictly $\preccurlyeq$-decreasing sequence. If addition and multiplication of such