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Expo. Math. 31 (2013) 385–391

Expositiones Mathematicae

www.elsevier.com/locate/exmath

A characterization of Euler's constant

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Received 10 December 2012; received in revised form 19 February 2013

Abstract

We prove the following theorem. Let α and β be real numbers. The inequality

$$\Gamma(x^\alpha + y^\beta) \le \Gamma\Big(\Gamma(x) + \Gamma(y)\Big)$$

holds for all positive real numbers x and y if and only if $\alpha = \beta = -\gamma$. Here, Γ and $\gamma = 0.57721\ldots$ denote Euler's gamma function and Euler's constant, respectively.

MSC 2010: primary 33B15; secondary 39B62

Keywords: Euler's constant; Gamma function; Functional inequalities

1. Introduction

The classical gamma function, introduced by L. Euler in 1729, is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \frac{1}{x} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{1}{k}\right)^x \left(1 + \frac{x}{k}\right)^{-1} \right\}.$$

In view of its importance in many mathematical branches as well as in related fields, the Γ-function has been the subject of intensive research. The main properties of the gamma function and its relatives are collected in, for instance, [1, Chapter 6]. Remarkable historical comments on this subject can be found in [2,3,6,7,11]. We also refer the reader to Sándor's detailed bibliography on the gamma function; see [10].

Our work has been motivated by an interesting research paper published by Monreal and Tomás [9] in 1998. In this article, the authors study several functional equations (in

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0723-0869/$ - see front matter
http://dx.doi.org/10.1016/j.exmath.2013.06.002

one and two variables) arising in computer graphics. One of these equations is

$$f(x+y) = f\big(f(x) + f(y)\big).$$

We investigate the functional inequality

$$\Gamma(x^{\alpha} + y^{\beta}) \le \Gamma\big(\Gamma(x) + \Gamma(y)\big). \tag{1.1}$$

More precisely, we ask for all real parameters α and β such that (1.1) is valid for all positive numbers x and y. It turns out that the answer to this question leads to a new characterization of the famous Euler constant γ. Indeed, in Section 3 we show that (1.1) is valid for all $x, y > 0$ if and only if $\alpha = \beta = -\gamma$.

The constant γ, introduced by Euler in 1734, is defined by the limit

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721\ldots.$$

It appears in several mathematical areas like analysis and number theory. Numerous series and integral representations for γ are known in the literature. A famous open problem is to prove that γ is an irrational number. The connection between the gamma function and γ is given by the formula $\Gamma'(1) = -\gamma$. Much information on Euler's constant can be found in the survey paper [4] and in the monographs [5,8].

The numerical values in this paper have been calculated via the computer program Maple V, Release 5.1.

2. Lemmas

Throughout this paper, we denote by $x_0 = 1.46163\ldots$ the only positive zero of $\psi = \Gamma'/\Gamma$. In order to prove our main result we need some inequalities for the gamma function. These inequalities are given in the following three lemmas.

Lemma 1. *For all $x > 0$ we have*

$$x^{-\gamma} \le \Gamma(x) \tag{2.1}$$

with equality holding if and only if $x = 1$.

Proof. We define for $x > 0$

$$g(x) = \log\Gamma(x) + \gamma\log x.$$

Then, we obtain

$$g'(x) = \psi(x) + \frac{\gamma}{x}.$$

Using the integral representation

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^{\infty} e^{-xt}\frac{t^n}{1-e^{-t}}dt \quad (n \in \mathbf{N};\, x > 0)$$