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An alternative approach to the structure theory of PI-rings☆

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ABSTRACT

We expose a rather simple and direct approach to the structure theory of prime PI-rings ("Posner's theorem"), based on fundamental properties of the extended centroid of a prime ring.

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1. Introduction

The theory of rings with polynomial identities originated in Kaplansky's 1948 paper [6], in which he showed that a primitive PI-algebra is finite dimensional over its center. In 1960 Posner [10] extended this theorem to the prime ring context; he proved that a prime PI-ring has a two-sided classical ring of quotients which is a finite-dimensional central simple algebra. After the discovery of central polynomials on matrix algebras in the early 1970s, Posner's theorem was further improved by noticing (by different authors, cf. [12]) that this ring of quotients is actually the algebra of central quotients.

A standard proof of this sharpened version of Posner's theorem, which can be found in several graduate algebra textbooks (e.g., in [1,9,13]), is a beautiful illustration of the power and applicability of the classical structure theory of rings. Its main ingredients are the Jacobson density theorem, the theorem by Nakayama and Azumaya on maximal subfields of division algebras, Amitsur's theorem on the Jacobson radical of the polynomial ring, the nonexistence of nonzero nil ideals in semiprime PI-rings, and the existence of central polynomials on matrices. The first two theorems are needed for the proof of Kaplansky's theorem, which is an intermediate step in this standard proof of Posner's theorem.

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The purpose of this paper is to give a more streamlined proof, which in each of its steps avoids representing elements in our rings as matrices or linear operators. All aforementioned ingredients are replaced by a single theorem (Theorem 2.1) describing one of the basic properties of the extended centroid of a prime ring. This theorem is essentially due to Martindale [8], and is one of the cornerstones in the theory of generalized polynomial identities [2] as well as in the theory of functional identities [4]. Our proof is in fact more typical for these two theories than for the PI theory. We have to point out, however, that the idea to use such an approach is not entirely new. Already in [8] Martindale noticed that Posner's theorem can be derived from his result on generalized polynomial identities in prime rings (see also [2]). But the proof of the latter is not so easy. Focusing only on polynomial identities, but abstractly regarding them as generalized polynomial and functional identities, we will be able to get a rather simple and straightforward proof.

Section 2 surveys the prerequisites needed for our proof. In Section 3 we study identities in central simple algebras, and in particular prove Kaplansky's theorem for them. This weak version of Kaplansky's theorem is our intermediate step which, as we show in Section 4, quickly yields the final result.

2. Preliminaries

The purpose of this section is to make this paper accessible to non-specialists. It is divided into two parts. In the first one we review the properties of the extended centroid and related notions, and in the second one we give an elementary introduction to polynomial identities.

By a ring we shall mean an associative ring, not necessarily having a unity 1. By an ideal we mean a two-sided ideal.

2.1. The extended centroid

Let R be a *prime ring*, i.e., a ring in which the product of two nonzero ideals is always nonzero. Then one can construct the *symmetric Martindale ring of quotients* $Q = Q_s(R)$ of R, which is, up to isomorphism, characterized by the following four properties:

(a) R is a subring of Q;
(b) for every $q \in Q$ there exists a nonzero ideal I of R such that $qI \cup Iq \subseteq R$;
(c) if I is a nonzero ideal of R and $0 \neq q \in Q$, then $qI \neq 0$ and $Iq \neq 0$;
(d) if I is a nonzero ideal of R, $f : I \to R$ is a right R-module homomorphism, and $g : I \to R$ is a left R-module homomorphism such that $xf(y) = g(x)y$ for all $x, y \in I$, then there exists $q \in Q$ such that $f(y) = qy$ and $g(x) = xq$ for all $x, y \in I$.

For details and some illustrative examples we refer the reader to [2,7]. We will be primarily interested in the center C of Q, called the *extended centroid* of R. It is a field containing the center Z of R. We remark that Z has no zero divisors, and therefore, provided it is nonzero, one can form its field of fractions. This is a subfield of C; examples for the case where it is a proper subfield can be easily constructed. Moreover, Z itself can be a field, but still a proper subfield of C. For example, if R is a ring of all countably infinite complex matrices of the form $A+\lambda$, where A is a matrix with only finitely many nonzero entries and λ is a real scalar matrix, then $Z \cong \mathbb{R}$ and $C \cong \mathbb{C}$.

We may consider Q as an algebra over C. A subalgebra of special importance is the so-called *central closure* of R, which we denote by R_C. It consists of elements of the form $\sum_i \lambda_i r_i$, where $\lambda_i \in C$ and $r_i \in R$. Both Q and R_C are prime rings. The extended centroid of R_C is just C. The same is true for every nonzero ideal of R_C (as well as of R). If $C \subseteq R_C$, then C is the center of R_C.

The main property of C that we need is given in the following theorem. Its original version was proved by Martindale in [8]. The version that we state is, as one can see from [4, Theorem A.4], a special case of [4, Theorem A.7].

Theorem 2.1. *Let R be a prime ring with extended centroid C, and let I be a nonzero ideal of R. Assume that $a_i, b_i, c_j, d_j \in Q_s(R)$ satisfy $\sum_{i=1}^{n} a_i x b_i = \sum_{j=1}^{m} c_j x d_j$ for all $x \in I$. If $a_1, \dots, a_n$ are linearly independent over C, then each b_i is a linear combination of $d_1, \dots, d_m$.*

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