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Power sums of polynomials over finite fields and applications: A survey ☆

Dinesh S. Thakur

University of Rochester, Rochester, NY 14627, United States

A R T I C L E I N F O

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A B S T R A C T

In this brief expository survey, we explain some results and conjectures on various aspects of the study of the sums of integral powers of monic polynomials of a given degree over a finite field. The aspects include non-vanishing criteria, formulas and bounds for degree and valuation at finite primes, explicit formulas of various kind for the sums themselves, factorizations of such sums, generating functions for them, relations between them, special type of interpolations of the sums by algebraic functions, and the resulting connections between the motives constructed from them and the zeta and multizeta special values. We mention several applications to the function field arithmetic.

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E-mail address: dinesh.thakur@rochester.edu.

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1. Introduction

In this expository survey, we explain some results, conjectures and applications of various aspects of the study of the sums of integral powers of monic polynomials of a given degree over a finite field.

The aspects include non-vanishing criteria, formulas and bounds for the degree and the valuation at finite primes for these sums, explicit formulas of various kind for the sums themselves, factorizations of such sums, generating functions for them, relations between them, special type of interpolations (Anderson's 'solitons' [36,35]) of the sums by algebraic functions, and the resulting connections between the motives constructed from them and the zeta and multizeta special values.

The combinatorics of cancellations can be very complicated, especially for non-prime finite fields. Thus there are many open questions and we mention several observations, guesses and conjectures.

After this, we mention and refer to several applications to several areas of function field arithmetic [16,29,36] (there are, of course, many other applications which we do not cover). These include evaluations of and relations between the special values of zeta and multizeta, understanding of the zero distribution of Riemann hypothesis type for the Goss zeta function, both at infinite and finite primes, study of variation in the p-ranks of Jacobians of cyclotomic coverings.

We do not give proofs (sometimes just mentioning the key idea) and give only the convenient references, where the details can be found. In describing the applications, we sometimes do not give the definitions of all the objects, referring to the cited papers for them and hoping that the description would still help the readers with general background on number theory.

2. Power sums

2.1. Notation

$$
\begin{aligned}
\mathbb{Z} &= \{\text{integers}\},\\
\mathbb{Z}_+ &= \{\text{positive integers}\},\\
\mathbb{Z}_{\geq 0} &= \{\text{nonnegative integers}\},\\
q &= \text{a power of a prime } p, \quad q = p^f,\\
A &= \mathbb{F}_q[t],\\
A+ &= \{\text{monics in } A\},\\
A_d+ &= \{\text{monics in } A \text{ of degree } d\},\\
A_{<d}+ &= \{\text{monics in } A \text{ of degree less than } d\},\\
K &= \mathbb{F}_q(t),
\end{aligned}
$$