Journal of Algebra 473 (2017) 29–65

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Minimal graded Lie algebras and representations of quadratic algebras

Hubert Rubenthaler

Institut de Recherche Mathématique Avancée, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France

ARTICLE INFO

Article history:
Received 14 December 2015
Available online 21 October 2016
Communicated by Alberto Elduque

MSC:
17B70
17B20

Keywords:
Graded Lie Algebras
Representations
Reductive Lie algebras
Quadratic Lie algebras

ABSTRACT

Let $(\mathfrak{g}_0, B_0)$ be a quadratic Lie algebra (i.e. a Lie algebra $\mathfrak{g}_0$ with a non-degenerate symmetric invariant bilinear form B_0) and let (ρ, V) be a finite dimensional representation of $\mathfrak{g}_0$. We define on $\Gamma(\mathfrak{g}_0, B_0, V) = V^* \oplus \mathfrak{g}_0 \oplus V$ a structure of local Lie algebra in the sense of Kac ([4]), where the bracket between $\mathfrak{g}_0$ and V (resp. V^*) is given by the representation ρ (resp. ρ^*), and where the bracket between V and V^* depends on B_0 and ρ. This implies the existence of two $\mathbb{Z}$-graded Lie algebras $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V))$ and $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ whose local part is $\Gamma(\mathfrak{g}_0, B_0, V)$. We investigate these graded Lie algebras, more specifically in the case where $\mathfrak{g}_0$ is reductive. Roughly speaking, the map $(\mathfrak{g}_0, B_0, V) \longmapsto \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is a bijection between triplets and a class of graded Lie algebras. We show that the existence of "associated $\mathfrak{sl}_2$-triples" is equivalent to the existence of non-trivial relative invariants on some orbit, and we define the "graded Lie algebras of polynomial type" which give rise to some dual airs.

Contents

E-mail address: rubenth@math.unistra.fr.

http://dx.doi.org/10.1016/j.jalgebra.2016.10.011
0021-8693/

1. Introduction

In this paper, following [4], a graded Lie algebra is a Lie algebra $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\, \mathfrak{g}_i$, such that $\dim \mathfrak{g}_i < +\infty$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for all $i, j \in \mathbb{Z}$, and such that $\mathfrak{g}$ is generated by its *local part* $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

If $\mathfrak{g} = \oplus_{i=-n}^{n}\mathfrak{g}_i$ is a grading of a (finite dimensional) complex semi-simple Lie algebra, it is well known that if B denotes the Killing form of $\mathfrak{g}$, then $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i + j \neq 0$. This allows us to identify $\mathfrak{g}_{-1}$ with the dual $\mathfrak{g}_1^*$. Moreover as B is invariant the bracket representation $(\mathfrak{g}_0, \mathfrak{g}_{-1})$ can be identified with the dual representation $(\mathfrak{g}_0, \mathfrak{g}_1^*)$.

It is then a natural question to ask if any finite dimensional representation $(\mathfrak{g}_0, \rho, V)$ of a finite dimensional Lie algebra $\mathfrak{g}_0$ can be embedded in a graded Lie algebra $\mathfrak{g} = \oplus_{i=-\infty}^{+\infty}\mathfrak{g}_i$ such that $(\mathfrak{g}_0, \mathfrak{g}_1) \simeq (\mathfrak{g}_0, \rho, V)$ and $(\mathfrak{g}_0, \mathfrak{g}_{-1}) \simeq (\mathfrak{g}_0, \rho^*, V^*)$, and such that the bracket between V and V^* is non-trivial.

The first result of this paper is to give a positive answer to this question for any representation of a quadratic Lie algebra. A quadratic Lie algebra is a pair $(\mathfrak{g}_0, B_0)$ where $\mathfrak{g}_0$ is a Lie algebra and B_0 a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}_0$. Of course the definition of the bracket between V and V^* will depend on B_0 and ρ.

We will use a result of V. Kac ([4]) which asserts that in order to construct a graded Lie algebra $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ it suffices to construct the local part $\Gamma = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, which has to be endowed with a partial Lie bracket (see section 2 for details). Therefore once we have build the partial bracket on the local part $\Gamma(\mathfrak{g}_0, B_0, \rho) = V^* \oplus \mathfrak{g}_0 \oplus V$, the existence of the "global" Lie algebra is just an application of a result of Kac. In fact Kac theory provides us with two such graded Lie algebras: a maximal one (denoted here $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho))$) and a minimal one (denoted $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$). Any graded Lie algebra with a given local part is a quotient of the maximal algebra, and has a