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Semi-invariant pictures and two conjectures on maximal green sequences

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ABSTRACT

We use semi-invariant pictures to prove two conjectures about maximal green sequences. First: if Q is any acyclic valued quiver with an arrow $j \to i$ of infinite type then any maximal green sequence for Q must mutate at i before mutating at j. Second: for any quiver Q' obtained by mutating an acyclic valued quiver Q of tame type, there are only finitely many maximal green sequences for Q'. Both statements follow from the Rotation Lemma for reddening sequences and this in turn follows from the Mutation Formula for the semi-invariant picture for Q.

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Introduction

Maximal green sequences are maximal paths in the oriented cluster exchange graph: Fixing an initial seed induces an orientation on the cluster exchange graph, a "green" sequence is an oriented path (passing an arrow from source to target is called "green", whereas passing an arrow in reverse direction is called "red"), and a maximal green sequence is one starting from the initial seed (the only source in the oriented exchange graph) to the unique sink (see Figs. 5 and 6). More generally, any sequence ending in the unique sink of an oriented cluster exchange graph is called a reddening sequence [9]. Categorification of cluster algebras [4] has led to a wealth of different interpretations and generalizations of the oriented cluster exchange graph, for instance as poset of functorially finite torsion classes, or of certain t-structures in a triangulated category, see [3] for an overview. In these poset interpretations, a maximal green sequence is simply a maximal chain. Following work of Reineke, Keller studied maximal green sequences to obtain quantum dilogarithm identities [9]. Moreover, maximal green sequences are considered in physics (under the name "finite chambers") when studying the BPS spectrum of a quantum field theory with extended supersymmetry, see [1,16] and references therein.

This paper proves two conjectures about maximal green sequences:

Theorem 1 *(Target before Source Conjecture). Given an acyclic valued quiver Q with an arrow $j \xrightarrow{(d_{ji},d_{ij})} i$ of infinite type, i.e., with $d_{ij}d_{ji} \geq 4$, any maximal green sequence mutates at the target i before the source j.*

Theorem 2 *(Finiteness Conjecture). If the valued quiver Q is mutation equivalent to an acyclic quiver of tame type, then Q has only finitely many maximal green sequences.*

Oriented cluster exchange graphs are associated to cluster algebras; the edges in the cluster exchange graph represent mutations, the fundamental notion in the definition of cluster algebras. The orientation of each edge indicates mutation in the direction of positive c-vectors.

The original idea of the proof of the Target before Source Conjecture came from the semi-invariant pictures of [11]. Using the fact that the lines are labeled by c-vectors and the normal orientation on the lines determines the sign of the c-vector, green mutations can be visualized as crossing the lines always in the direction of the normal orientation as illustrated in Fig. 1.

A maximal green sequence is a path going from the outside, unbounded region to the center which only goes inward at each wall. The double arrow $3 \rightrightarrows 2$ creates infinite families of walls. The solid line is the maximal green sequence $(2, 1, 3, 2)$ and the dotted lines are green sequences which cannot be extended to maximal green sequences showing that maximal green sequences cannot mutate 3 before 2. Here the integer k at each step indicates the mutation μ_k in the direction of the k-th c-vector, which is the same as mutation at vertex k of the quiver Q defining the cluster algebra.