Journal of Algebra 473 (2017) 132–141

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Vanishing ranges for the mod $p$ cohomology of alternating subgroups of Coxeter groups

Toshiyuki Akita *, Ye Liu

*Department of Mathematics, Hokkaido University, Sapporo, 060-0810 Japan*

ARTICLE INFO

*Article history:*
Received 27 April 2016
Available online 9 November 2016
Communicated by Michel Broué

*MSC:*
primary 20F55, 20J06
secondary 55N91

*Keywords:*
Group cohomology
Coxeter groups
Alternating subgroups

ABSTRACT

We obtain vanishing ranges for the mod $p$ cohomology of alternating subgroups of finite $p$-free Coxeter groups. Here a Coxeter group $W$ is $p$-free if the order of the product $st$ is prime to $p$ for every pair of Coxeter generators $s, t$ of $W$. Our result generalizes those for alternating groups formerly proved by Kleshchev–Nakano and Burichenko. As a byproduct, we obtain vanishing ranges for the twisted cohomology of finite $p$-free Coxeter groups with coefficients in the sign representations. In addition, a weak version of the main result is proved for a certain class of infinite Coxeter groups.



## 1. Introduction

Let $(W, S)$ be a Coxeter system, the pair of a Coxeter group $W$ and the set $S$ of Coxeter generators of $W$. The alternating subgroup $A_W$ of $W$ is the kernel of the sign homomorphism $W \to \{\pm 1\}$ defined by $s \mapsto -1$ $(s \in S)$. The symmetric group $\Sigma_n$ on $n$ letters is a finite Coxeter group (of type $A_{n-1}$), and its alternating subgroup is nothing but the alternating group $A_n$ on $n$ letters. Cohomology of alternating groups has been studied by various authors. Among others, Kleshchev–Nakano [9, p. 354 Corollary] and

* Corresponding author.
*E-mail addresses:* akita@math.sci.hokudai.ac.jp (T. Akita), liu@math.sci.hokudai.ac.jp (Y. Liu).

http://dx.doi.org/10.1016/j.jalgebra.2016.11.005

Burichenko [7, Theorem 1.4] independently obtained vanishing ranges for the mod $p$ cohomology of alternating groups:

**Theorem 1.1** *(Kleshchev–Nakano, Burichenko). Let $p$ be an odd prime. The mod $p$ cohomology $H^k(A_n, \mathbb{F}_p)$ vanishes for $0 < k < p - 2$.*

The primary purpose of this paper is to generalize Theorem 1.1 to alternating subgroups of finite Coxeter groups, and thereby to give an alternative proof of Theorem 1.1. Our main result is the following:

**Theorem 1.2.** *Let $p$ be an odd prime, $W$ a finite $p$-free Coxeter group, and $A_W$ the alternating subgroup of $W$. Then the mod $p$ cohomology $H^k(A_W, \mathbb{F}_p)$ vanishes for $0 < k < p - 2$.*

Here a Coxeter group is $p$-free if the order of the product $st \in W$ is prime to $p$ for every pair of Coxeter generators $s, t \in S$. Since symmetric groups $\Sigma_n$ are $p$-free for $p \geq 5$ as Coxeter groups, Theorem 1.1 is a special case of our theorem. Note that finiteness and $p$-freeness assumptions on $W$ are necessary, and vanishing ranges for $H^k(A_W, \mathbb{F}_p)$ are best possible. See §2.2, §6 and §7 below.

The key ingredients for the proof of Theorem 1.2 are

(1) the classification of finite Coxeter groups (see §3),
(2) high connectivity of the Coxeter complex $X_W$ (Proposition 5.1, 5.2),
(3) high connectivity of the orbit space $X_W/A_W$ (Proposition 5.3),

as well as some considerations of equivariant cohomology in §4. The proof is inspired by the arguments in [2], where the first author obtained vanishing ranges for the $p$-local homology of $p$-free Coxeter groups. As a byproduct of Theorem 1.2, we will obtain vanishing ranges for the twisted cohomology of finite $p$-free Coxeter groups with coefficients in the sign representations over $\mathbb{F}_p$ (Theorem 6.1). As we remarked above, Theorem 1.2 no longer holds for infinite Coxeter groups. Instead, we will prove a weak version of Theorem 1.2 for a certain class of infinite Coxeter groups (Theorem 7.2).

## 2. Preliminaries

### *2.1. Coxeter groups*

In this subsection, we recall definitions and relevant facts concerning of Coxeter groups. References are [1,4,8]. Let $S$ be a finite set. A Coxeter matrix is a symmetric matrix $M = (m(s,t))_{s,t \in S}$ each of whose entries $m(s,t)$ is a positive integer or $\infty$ such that

(1) $m(s,s) = 1$ for all $s \in S$,
(2) $2 \leq m(s,t) = m(t,s) \leq \infty$ for all distinct $s, t \in S$.