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Graded cellularity and the monotonicity conjecture

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ABSTRACT

The graded cellularity of Libedinsky double leaves, which form a basis for the endomorphism ring of the Bott–Samelson–Soergel bimodules, allows us to view the Kazhdan–Lusztig polynomials as graded decomposition numbers. Using this interpretation, we can provide a proof of the monotonicity conjecture for any Coxeter system.

1. Introduction

In their seminal paper [12], Kazhdan and Lusztig defined, for each Coxeter system (W, S), a family of polynomials with integer coefficients indexed by pairs of elements of W. These polynomials are now known as the Kazhdan–Lusztig (KL) polynomials. We will denote them by $P_{x,w}(q) \in \mathbb{Z}[q]$, for all $x, w \in W$. Applications of the KL-polynomials have been found in the representation theory of semisimple algebraic groups, the topology of Schubert varieties, the theory of Verma modules, the Bernstein–Gelfand–Gelfand (BGG) category $\mathcal{O}$, etc. (see, e.g., [2] and references therein).

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Aside from the importance of the KL-polynomials in the above-mentioned subjects, there are purely combinatorial reasons to study these polynomials. Perhaps the major reason is the longstanding Kazhdan–Lusztig positivity conjecture [12], which states that $P_{x,w}(q) \in \mathbb{N}[q]$, for all Coxeter groups W and all $x, w \in W$. In 2014, Elias and Williamson [6] gave a proof of this conjecture by proving a stronger result known as Soergel's conjecture.

For each Coxeter group W, Soergel constructed a category of graded R-bimodules (where R is a polynomial ring with coefficients in $\mathbb{R}$) known as the category of Soergel bimodules, which we will denote by $\mathbb{S}\mathrm{Bim}$. He proved that (up to degree shift) W parameterizes the set of all indecomposable objects in $\mathbb{S}\mathrm{Bim}$. For $w \in W$, let us denote by B_w the corresponding indecomposable object. Soergel proved in [18] that $\mathbb{S}\mathrm{Bim}$ is a categorification of the Hecke algebra $\mathcal{H}$ of W. This means that there is an algebra isomorphism

$$\epsilon : \mathcal{H} \to [\mathbb{S}\mathrm{Bim}], \tag{1.1}$$

where $[\mathbb{S}\mathrm{Bim}]$ denotes the split Grothendieck group of $\mathbb{S}\mathrm{Bim}$. Soergel proposed the following conjecture, which came to be known as Soergel's conjecture:

$$\epsilon(\underline{H}_w) = [B_w], \tag{1.2}$$

where $\{\underline{H}_w\}_{w\in W}$ is the Kazhdan–Lusztig basis of $\mathcal{H}$. Assuming this conjecture, Soergel showed that the Kazhdan–Lusztig polynomials of W arise as graded ranks of Hom spaces between indecomposable Soergel bimodules and standard bimodules. It follows that these coefficients are non-negative, i.e., he proved that (1.2) implies the positivity conjecture.

More than mere positivity is conceivable for coefficients of KL-polynomials. In effect, a monotonicity property is known for these coefficients, when W is a finite or affine Weyl group. Namely, for these groups it is true that if $u, v, w \in W$ and $u \leq v \leq w$, then

$$P_{u,w}(q) - P_{v,w}(q) \in \mathbb{N}[q], \tag{1.3}$$

where $\leq$ denotes the usual Bruhat order on W. In other words, if we fix the second index of a KL-polynomial, and if the first one decreases in Bruhat order, all coefficients in the polynomial weakly increase in value. This result was originally proved by Irving [10, Corollary 4] using the interpretation of KL-polynomials as multiplicities of simple objects in the socle filtration of a Verma module, when W is a finite Weyl group, and by Braden and MacPherson [3, Corollary 3.7] using the interpretation of KL-polynomials as Poincaré polynomials of the local intersection cohomology of Schubert varieties, when W is a finite or affine Weyl group.

It is natural to conjecture that (1.3) holds for arbitrary Coxeter groups. In the literature, the latter conjecture is referred to as the *Monotonicity Conjecture*[2] for KL-

[2] To the best of the author's knowledge nobody conjectured the Monotonicity Conjecture.