Journal of Algebra 473 (2017) 374–396

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Global Oort groups ☆

Ted Chinburg [a], Robert Guralnick [b,*], David Harbater [a]

[a] Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA
[b] Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA

ARTICLE INFO

Article history:
Received 31 December 2015
Available online 10 November 2016
Communicated by Gunter Malle

MSC:
primary 20B25, 12F10, 14H37
secondary 13B05, 14D15, 14H30

Keywords:
Curves
Covers
Automorphisms
Galois groups
Characteristic p
Lifting
Oort conjecture
Simple group
Almost simple
Normal complement

ABSTRACT

We study the Oort groups for a prime p, i.e. finite groups G such that every G-Galois branched cover of smooth curves over an algebraically closed field of characteristic p lifts to a G-cover of curves in characteristic 0. We prove that all Oort groups lie in a particular class of finite groups that we characterize, with equality of classes under a conjecture about local liftings. We prove this equality unconditionally if the order of G is not divisible by $2p^2$. We also treat the local lifting problem and relate it to the global problem.

☆ The authors were supported by NSF FRG grant DMS-1265290. The first author was also supported by NSF FRG DMS-1360767, NSF SaTC grant CNS-1513671 and Simons Foundation Grant 338379. The second author was also supported by NSF DMS-1265297 and NSF DMS-1302886. The third author was also supported by NSA grant H98230-14-1-0145 and NSF FRG grant DMS-1463733.
* Corresponding author.
E-mail addresses: ted@math.upenn.edu (T. Chinburg), guralnic@usc.edu (R. Guralnick), harbater@math.upenn.edu (D. Harbater).

http://dx.doi.org/10.1016/j.jalgebra.2016.10.034
0021-8693/

1. Introduction

This paper, which is a sequel to [7] and [8], concerns the question of when covers of curves in characteristic $p > 0$ lift to covers in characteristic zero. We begin with a G-Galois finite branched cover $X \to Y$ of smooth projective curves over an algebraically closed field k of characteristic p. By a *lifting* we mean a G-Galois branched cover $\mathcal{X} \to \mathcal{Y}$ of normal projective curves over a complete discrete valuation ring R such that the following is true. The fraction field of R has characteristic zero, the residue field of R is isomorphic to k, and the closed fiber of $\mathcal{X} \to \mathcal{Y}$ is G-isomorphic to the given cover $X \to Y$. Lifting a G-Galois cover $X \to Y$ is equivalent to lifting the action of G on X to an action of G on $\mathcal{X}$.

In [13], Grothendieck showed that all tamely ramified covers lift, and in particular a cover lifts if the Galois group has order prime to p. Later, in [18, Sect. I.7], F. Oort posed the following conjecture:

Conjecture 1.1 *(Oort Conjecture). Every cyclic Galois cover of k-curves lifts to characteristic zero.*

Motivated by this conjecture, in [7] we defined a finite group G to be a (global) *Oort group* for k if every G-Galois cover of smooth projective k-curves lifts to characteristic zero. We asked which finite groups are Oort groups. Recently, Conjecture 1.1 was proven by the combined work of Obus and Wewers [17] and Pop [21], so we now know that every cyclic group is an Oort group. A brief history is as follows: In [19] and later in [12], it was shown that a cyclic group G is an Oort group provided that its order is exactly divisible by p, respectively by p^2. More recently, in [17], it was shown that a cyclic group G is an Oort group provided that its order is divisible at most by p^3, or more generally if its higher ramification groups satisfy a certain condition. Finally, by deforming a general cover to one in the special case treated in [17], the full conjecture was proven in [21].

While the above conjecture considered only cyclic groups, some results have been obtained about more general groups. Apart from Grothendieck's result on prime-to-p groups being Oort groups, it is known that the dihedral group D_{2p} of order $2p$ is an Oort group for algebraically closed fields of characteristic p (see [20] for $p = 2$, [4] for p odd), and A_4 is an Oort group for algebraically closed fields of characteristic 2 (see [4] and [16]). Note that these groups are cyclic-by-p, i.e. they are extensions of a cyclic group of order prime to p by a normal Sylow p-subgroup. The significance of understanding Oort groups of this form is that a group G is an Oort group if and only if each cyclic-by-p subgroup of G is an Oort group [7, Corollary 2.8].

In [7, Corollary 3.4, Theorem 4.5], we proved that if a cyclic-by-p group G is an Oort group for an algebraically closed field of characteristic p, then G is either a cyclic group C_m of order m, a dihedral group D_{2p^n} for some n, or A_4 if $p = 2$. We also conjectured the converse in [7]: