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Action of special linear groups to the tensor of indeterminates and classical invariants of binary forms

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ABSTRACT

In this paper, we study the ring of invariants under the action of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$ and $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K) \times \mathrm{SL}(2, K)$ on the 3-dimensional tensor of indeterminates of form $m \times n \times 2$, where K is an infinite field. We show that if $m = n \geq 2$, then the ring of $\mathrm{SL}(n, K) \times \mathrm{SL}(n, K)$-invariants is generated by $n+1$ algebraically independent elements over K and the action of $\mathrm{SL}(2, K)$ on that ring is identical with the one defined in the classical invariant theory of binary forms. We also reveal the ring of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$-invariants and $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K) \times \mathrm{SL}(2, K)$-invariants completely in the case where $m \neq n$.

1. Introduction

High-dimensional array data analysis is now rapidly developing and being successfully applied in various fields. A high-dimensional array datum is called a tensor in those

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communities. To be precise, a d-dimensional array datum $(a_{i_1 i_2 \cdots i_d})_{1 \leq i_j \leq m_j}$ is called a d-tensor or an $m_1 \times \cdots \times m_d$-tensor.

A 2-tensor is no other than a matrix. For a matrix of indeterminates, that is, a matrix whose entries are independent indeterminates, there are many results about the action of various subgroups of the general linear group and the rings of invariants under this action.

To be precise, let $X = (X_{ij})$ be an $m \times n$ matrix of indeterminates, that is, a matrix whose entries are independent indeterminates, and G a subgroup of $\mathrm{GL}(m, K)$, where K is an infinite field. For $P \in G$, one defines the action of P on $K[X] = K[X_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n]$ by the K algebra homomorphism sending X to $P^\top X$. Let us state this in other words. Let V_1 and V_2 be vector spaces over K with dimensions m and n respectively. Since G acts on V_1 linearly, one can extend it to the action of G on $\mathrm{Sym}(V_1 \otimes V_2)$, the symmetric algebra of $V_1 \otimes V_2$ over K.

In this paper, we first consider the action of the product of two special linear groups on a 3-tensor of indeterminates, that is, a 3-tensor whose entries are independent indeterminates. Let $T = (T_{ijk})_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2}$ be an $m \times n \times 2$-tensor of indeterminates. Set $X_k = (T_{ijk})_{1 \leq i \leq m, 1 \leq j \leq n}$ for $k = 1, 2$. Then $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$ acts on the polynomial ring $K[T] = K[T_{ijk} \mid 1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq 2]$ by the K-algebra homomorphism sending X_k to $P^\top X_k Q$ for $k = 1, 2$, where $(P, Q) \in \mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$.

To state it in other words, let V_1, V_2 and V_3 be K-vector spaces of dimensions m, n and 2 respectively. Then the natural actions of $\mathrm{SL}(m, K)$ on V_1 and $\mathrm{SL}(n, K)$ on V_2 induce an action of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$ on $\mathrm{Sym}(V_1 \otimes V_2 \otimes V_3)$. About this action, we show the following facts. (1) If $m = n \geq 2$, then $K[T]^{\mathrm{SL}(n,K) \times \mathrm{SL}(n,K)}$ is generated by $n + 1$ algebraically independent elements over K. (2) If $n = m + \gcd(m, n)$, then $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)}$ is generated by one element over K. (3) If $m < n$ and $n \neq m + \gcd(m, n)$, then $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)} = K$. In fact, we show that these generators are sagbi basis of the ring of invariants. (See Theorems 3.8 and 3.16.)

Next we consider the action of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K) \times \mathrm{SL}(2, K)$ on $K[T]$, in particular, the action of $\mathrm{SL}(2, K)$ on $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)}$. We show, above all things, that in the case where $m = n$ and $K = \mathbb{C}$, this action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathrm{Sym}(V_1 \otimes V_2 \otimes V_3)^{\mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C})} = \mathbb{C}[T]^{\mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C})}$ coincides with the action of classical invariant theory of binary forms. See Theorem 4.8. Since the theory of classical invariants dates back to nineteenth century, using the accumulated results on classical invariants of binary forms [12,8,6,10,7,1,2] and combining the results of this paper, one obtains much information about $\mathrm{SL}(n, \mathbb{C}) \times \mathrm{SL}(n, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$-invariant polynomials in $\{T_{ijk}\}_{1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq 2}$.

The organization of this paper is as follows. After establishing notation and recalling basic facts in Section 2, we study in Section 3 the invariants in $K[T]$ under the action of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$ stated above and show the above mentioned results. In Section 4, we study the invariants in $K[T]$ under the action of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K) \times \mathrm{SL}(2, K)$ and show the following facts. (1) If $m \neq n$, then $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K) \times \mathrm{SL}(2,K)} = K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)}$. (2) If $m = n$ and $K = \mathbb{C}$, then the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}[T]^{\mathrm{SL}(m,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C})}$ is isomorphic to the action of $\mathrm{SL}(2, \mathbb{C})$ considered in the classical