Journal of Algebra 473 (2017) 447–480

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Type A blocks of super category $\mathcal{O}$ ☆

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ARTICLE INFO

Article history:
Received 24 June 2016
Available online 5 December 2016
Communicated by Masaki Kashiwara

Keywords:
Lie superalgebras
Category $\mathcal{O}$

ABSTRACT

We show that every block of category $\mathcal{O}$ for the general linear Lie superalgebra $\mathfrak{gl}_{m|n}(\Bbbk)$ is equivalent to some corresponding block of category $\mathcal{O}$ for the queer Lie superalgebra $\mathfrak{q}_{m+n}(\Bbbk)$. This implies the truth of the Kazhdan–Lusztig conjecture for the so-called type A blocks of category $\mathcal{O}$ for the queer Lie superalgebra as formulated by Cheng, Kwon and Wang.

1. Introduction

In this article, we study the analog of the BGG category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\Bbbk)$. Recent work of Chen [11] has reduced most questions about this category just to the study of three particular types of block, which we refer to here as the type A, type B and type C blocks. Type B blocks (which correspond to integral weights) were investigated already by the first author in [2], leading to a Kazhdan–Lusztig conjecture for characters of irreducibles in such blocks in terms of certain canonical bases for the quantum group of type B_∞. In [14], Cheng, Kwon and Wang formulated analogous conjectures for the type A blocks (defined below) and the type C blocks (which correspond to half-integral weights) in terms of canonical bases of quantum groups of types A_∞ and C_∞, respectively.

☆ Research supported in part by NSF grant DMS-1161094.
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http://dx.doi.org/10.1016/j.jalgebra.2016.11.022
0021-8693/

The main goal of the article is to prove the Cheng–Kwon–Wang conjecture for type A blocks [14, Conjecture 5.14]. To do this, we use some tools from higher representation theory to establish an equivalence of categories between the type A blocks of category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\Bbbk)$ and integral blocks of category $\mathcal{O}$ for a general linear Lie superalgebra. This reduces the Cheng–Kwon–Wang conjecture for type A blocks to the Kazhdan–Lusztig conjecture of [1], which was proved already in [15,8].

Regarding the types B and C conjectures, Tsuchioka discovered in 2010 that the type B canonical bases considered in [2] fail to satisfy appropriate positivity properties, so that the conjecture from [2] is certainly false. Moreover, after the first version of [14] appeared, Tsuchioka pointed out similar issues with the type C canonical bases studied in [14], so that the Cheng–Kwon–Wang conjecture for type C blocks as formulated in the first version of their article [14, Conjecture 5.10] also seems likely to be incorrect.

In fact, the techniques developed in this article can be applied also to the study of the type C blocks. This will be spelled out in a sequel to this paper [4]. In this sequel, we prove a modified version of the Cheng–Kwon–Wang conjecture for type C blocks: one needs to replace Lusztig's canonical basis with Webster's "orthodox basis" arising from the indecomposable projective modules of the tensor product algebras of [29, §4]. This modified conjecture was proposed independently by Cheng, Kwon and Wang in a revision of their article [14, Conjecture 5.12]. It is not as satisfactory as the situation for type A blocks, since there is no elementary algorithm to compute Webster's basis explicitly (unlike the canonical basis). Also in the sequel, we will prove [14, Conjecture 5.13], and settle [14, Question 5.1] by identifying the category of finite-dimensional half-integer weight representations of $\mathfrak{q}_n(\Bbbk)$ with a previously known highest weight category (as suggested by [13, Remark 6.7]).

There is more to be said about type B blocks too; in fact, these are the most intriguing of all. Whereas the types A and C blocks carry the additional structure of tensor product categorifications in the sense of [24,8] for the infinite rank Kac–Moody algebras of types A_∞ and C_∞, respectively, the type B blocks produce an example of a tensor product categorification of *odd* type B_∞, i.e. one needs a *super* Kac–Moody 2-category in the sense of [6]. This will be developed in subsequent work by the second author.

In the remainder of the introduction, we are going to formulate our main result for type A blocks in more detail. To do this, we first briefly recall some basic notions of superalgebra. Let $\Bbbk$ be a ground field which is algebraically closed of characteristic zero, and fix a choice of $\sqrt{-1} \in \Bbbk$. We adopt the language of [5, Definition 1.1]:

- A *supercategory* is a category enriched in the symmetric monoidal category of vector superspaces, i.e. the category of $\mathbb{Z}/2$-graded vector spaces over $\Bbbk$ with morphisms that are parity-preserving linear maps.
- Any morphism in a supercategory decomposes uniquely into an even and an odd morphism as $f = f_{\bar{0}} + f_{\bar{1}}$. A *superfunctor* between supercategories means a $\Bbbk$-linear functor which preserves the parities of morphisms.