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Uniform measures on braid monoids and dual braid monoids

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ABSTRACT

We aim at studying the asymptotic properties of typical *positive braids*, respectively *positive dual braids*. Denoting by μ_k the uniform distribution on positive (dual) braids of length k, we prove that the sequence $(\mu_k)_k$ converges to a unique probability measure μ_∞ on *infinite* positive (dual) braids. The key point is that the limiting measure μ_∞ has a Markovian structure which can be described explicitly using the combinatorial properties of braids encapsulated in the Möbius polynomial. As a by-product, we settle a conjecture by Gebhardt and Tawn (J. Algebra, 2014) on the shape of the Garside normal form of large uniform braids.

1. Introduction

Consider a given number of strands, say n, and the associated positive braid monoid B_n^+ defined by the following monoid presentation, known as the *Artin* presentation:

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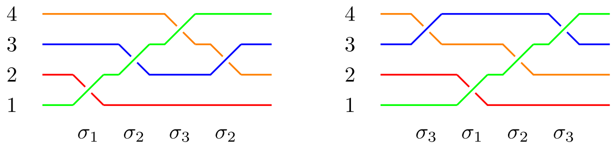

Fig. 1. Two isotopic braid diagrams representing a positive braid on 4 strands. Left: diagram corresponding to the word $\sigma_1\sigma_2\sigma_3\sigma_2$. Right: diagram corresponding to the word $\sigma_3\sigma_1\sigma_2\sigma_3$.

$$B_n^+ = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle^+ . \tag{1}$$

The elements of B_n^+, the *positive braids*, are therefore equivalence classes of words over the alphabet $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$. Alternatively, going back to the original geometric intuition, positive braids can be viewed as isotopy classes of *positive* braid diagrams, that is, braid diagrams in which the bottom strand always goes on top in a crossing, see Fig. 1.

We want to address the following question:

What does a typical complicated positive braid look like?

To make the question more precise, we need to clarify the meaning of "complicated" and "typical". First, let the complexity of a positive braid be measured by the length (number of letters) of any representative word. This is natural since it corresponds to the number of crossings between strings in any representative braid diagram. Therefore, a positive braid is "complicated" if its length is large.

Second, let us define a "typical" braid as a braid being picked at random according to some probability measure. The two natural candidates for such a probability measure are as follows. Fix a positive integer k.

- The first option consists in running a simple random walk on B_n^+: pick a sequence of random elements $x_i, i \geq 1$, independently and uniformly among the generators $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$, and consider the "typical" braid $X = x_1 \cdot x_2 \cdot \cdots \cdot x_k$. It corresponds to drawing a word uniformly in Σ^k and then considering the braid it induces.
- The second option consists in picking a "typical" braid of length k uniformly at random among all braids of length k.

The two approaches differ since the number of representative words varies among positive braids of the same length. For instance, in B_3^+ and for the length 3, the braid $\sigma_1 \cdot \sigma_2 \cdot \sigma_1$ $(= \sigma_2 \cdot \sigma_1 \cdot \sigma_2)$ will be picked with probability $2/8$ in the first approach, and with probability $1/7$ in the second one, while all the other braids of length 3 will be picked respectively with probabilities $1/8$ and $1/7$ in the two approaches. The random walk approach has been studied for instance in [26,30]; it is a special instance of random