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Commutative post-Lie algebra structures on Lie algebras ☆

Dietrich Burde *, Wolfgang Alexander Moens

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

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ABSTRACT

We show that any CPA-structure (commutative post-Lie algebra structure) on a perfect Lie algebra is trivial. Furthermore we give a general decomposition of inner CPA-structures, and classify all CPA-structures on complete Lie algebras. As a special case we obtain the CPA-structures of parabolic subalgebras of semisimple Lie algebras.

1. Introduction

Post-Lie algebras have been introduced by Valette in connection with the homology of partition posets and the study of Koszul operads [22]. Loday [17] studied pre-Lie algebras and post-Lie algebras within the context of algebraic operad triples. We rediscovered post-Lie algebras as a natural common generalization of pre-Lie algebras [13,15,20,4–6] and LR-algebras [8,9] in the geometric context of nil-affine actions of Lie groups. We

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* Corresponding author.
E-mail addresses: dietrich.burde@univie.ac.at (D. Burde), wolfgang.moens@univie.ac.at (W.A. Moens).

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then studied post-Lie algebra structures in general, motivated by the importance of pre-Lie algebras in geometry, and in connection with generalized Lie algebra derivations [7,10–12]. In particular, the existence question of post-Lie algebra structures on a given pair of Lie algebras turned out to be very interesting and quite challenging. But even if existence is clear the question remains how many structures are possible. In [12] we introduced a special class of post-Lie algebra structures, namely *commutative* ones. We conjectured that any commutative post-Lie algebra structure, in short CPA-structure, on a complex, perfect Lie algebra is *trivial*. For several special cases we already proved the conjecture in [12], but the general case remained open. One main result of this article here is a full proof of this conjecture, see Theorem 3.3. Furthermore we also study inner CPA-structures and give a classification of CPA-structures on parabolic subalgebras of semisimple Lie algebras.

In section 2 we study ideals of CPA-structures, non-degenerate and inner CPA-structures. In particular we show that any CPA-structure on a complete Lie algebra is inner. We give a general decomposition of inner CPA-structures, see Theorem 2.14. This implies, among other things, that any Lie algebra $\mathfrak{g}$ admitting a non-degenerate inner CPA-structure is metabelian, i.e., satisfies $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]=0$.

In section 3 we prove the above conjecture and generalize the result to perfect subalgebras of arbitrary Lie algebras in Theorem 3.4. This also implies that any Lie algebra admitting a non-degenerate CPA-structure is solvable. Conversely we show that any non-trivial solvable Lie algebra admits a non-trivial CPA-structure.

In section 4 we classify CPA-structures on complete Lie algebras satisfying a certain technical condition. As an application we obtain all CPA-structures on parabolic subalgebras of semisimple Lie algebras.

2. Preliminaries

Let K always denote a field of characteristic zero. Post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g},\mathfrak{n})$ over K are defined as follows [10]:

Definition 2.1. Let $\mathfrak{g}=(V,[\,,])$ and $\mathfrak{n}=(V,\{\,,\})$ be two Lie brackets on a vector space V over K. A *post-Lie algebra structure* on the pair $(\mathfrak{g},\mathfrak{n})$ is a K-bilinear product $x\cdot y$ satisfying the identities:

$$x\cdot y-y\cdot x=[x,y]-\{x,y\} \tag{1}$$

$$[x,y]\cdot z=x\cdot(y\cdot z)-y\cdot(x\cdot z) \tag{2}$$

$$x\cdot\{y,z\}=\{x\cdot y,z\}+\{y,x\cdot z\} \tag{3}$$

for all $x,y,z\in V$.

Define by $L(x)(y)=x\cdot y$ and $R(x)(y)=y\cdot x$ the left respectively right multiplication operators of the algebra $A=(V,\cdot)$. By (3), all $L(x)$ are derivations of the Lie algebra