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Rota–Baxter systems, dendriform algebras and covariant bialgebras

Tomasz Brzeziński [a,b,*]

[a] *Department of Mathematics, Swansea University, Swansea SA2 8PP, UK*
[b] *Department of Mathematics, University of Białystok, K. Ciołkowskiego 1M, 15-245 Białystok, Poland*

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ABSTRACT

A generalisation of the notion of a Rota–Baxter operator is proposed. This generalisation consists of two operators acting on an associative algebra and satisfying equations similar to the Rota–Baxter equation. Rota–Baxter operators of any weights and twisted Rota–Baxter operators are solutions of the proposed system. It is shown that dendriform algebra structures of a particular kind are equivalent to Rota–Baxter systems. It is shown further that a Rota–Baxter system induces a *weak pseudotwistor* [Panaite and Van Oystaeyen (2015) [15]] which can be held responsible for the existence of a new associative product on the underlying algebra. Examples of solutions of Rota–Baxter systems are obtained from quasitriangular covariant bialgebras hereby introduced as a natural extension of *infinitesimal bialgebras* [Aguiar (2000) [3]].

* Correspondence to: Department of Mathematics, Swansea University, Swansea SA2 8PP, UK.
E-mail address: T.Brzezinski@swansea.ac.uk.

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1. Introduction

This paper arose from an attempt to understand the Jackson q-integral as a Rota–Baxter operator, and develops and extends connections between three algebraic systems: Rota–Baxter algebras [16], dendriform algebras [13] and infinitesimal bialgebras [3].

Given an associative algebra A over a field $\mathbb{K}$ and $\lambda \in \mathbb{K}$, a linear operator $R : A \to A$ is called a *Rota–Baxter operator of weight* λ if, for all $a, b \in A$,

$$R(a)R(b) = R\left(R(a)b + aR(b) + \lambda ab\right). \tag{1.1}$$

In this case the triple (A, R, λ) is referred to as a *Rota–Baxter algebra of weight* λ. Rota–Baxter operators were introduced in [5] in the context of differential operators on commutative Banach algebras and since [16] intensively studied in probability and combinatorics, and more recently in the theory of operads and renormalisation of quantum field theories.

Introduced in [13, Section 5], a *dendriform algebra* is a system consisting of a vector space V and two bilinear operations $\prec$, $\succ$ on V such that, for all $a, b, c \in V$,

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \tag{1.2a}$$

$$a \succ (b \prec c) = (a \succ b) \prec c, \tag{1.2b}$$

$$a \succ (b \succ c) = (a \prec b + a \succ b) \succ c. \tag{1.2c}$$

As explained in [2,8,9] every Rota–Baxter operator of weight λ on an algebra A defines a dendriform algebra structure on A in a variety of ways including

$$a \succ b := R(a)b, \qquad a \prec b := a(R(b) + \lambda b). \tag{1.3}$$

An associative (not necessarily unital) algebra A that admits a coassociative comultiplication which is a derivation is called an *infinitesimal bialgebra* [3]. That is, in addition to the coassociative law, the comultiplication $\Delta : A \to A \otimes A$ satisfies

$$\Delta(ab) = a\Delta(b) + \Delta(a)b, \qquad \text{for all } a, b \in A, \tag{1.4}$$

where $A \otimes A$ is viewed as an A-bimodule in the standard way $a \cdot (b \otimes c) \cdot d = ab \otimes cd$. As shown in [3], if an element $r \in A \otimes A$ satisfies the associative classical Yang–Baxter equation,

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0 \tag{1.5}$$

(see (1.6) below for the explanation of the index notation used), then the inner derivation induced by r (i.e. a commutator with r) is coassociative, and thus defines on A the structure of an infinitesimal bialgebra (called a *quasitriangular infinitesimal bialgebra*). Furthermore, it is proven in [4] that every solution of the associative classical Yang–Baxter equation defines a Rota–Baxter operator of weight 0.