Journal of Algebra 460 (2016) 60–101

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Canonical complexes associated to a matrix

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A R T I C L E   I N F O

Article history:
Received 15 September 2015
Available online 4 May 2016
Communicated by Luchezar L. Avramov

MSC:
primary 13D02
secondary 13C40

Keywords:
Buchsbaum–Rim complex
Depth-sensitivity
Determinantal ring
Duality
Eagon–Northcott complex
Hooks
Koszul complex
Maximal Cohen–Macaulay module
Perfect module
Schur module
Weyl module

A B S T R A C T

Let Φ be an $\mathfrak{f}\times\mathfrak{g}$ matrix with entries from a commutative Noetherian ring R, with $\mathfrak{g}\le\mathfrak{f}$. Recall the family of generalized Eagon–Northcott complexes $\{\mathcal{C}^i_\Phi\}$ associated to Φ. (See, for example, Appendix A2 in "Commutative Algebra with a View Toward Algebraic Geometry" by D. Eisenbud.) For each integer i, $\mathcal{C}^i_\Phi$ is a complex of free R-modules. For example, $\mathcal{C}^0_\Phi$ is the original "Eagon–Northcott" complex with zero-th homology equal to the ring $R/I_\mathfrak{g}(\Phi)$ defined by ideal generated by the maximal order minors of Φ; and $\mathcal{C}^1_\Phi$ is the "Buchsbaum–Rim" complex with zero-th homology equal to the cokernel of the transpose of Φ. If Φ is sufficiently general, then each $\mathcal{C}^i_\Phi$, with $-1\le i$, is acyclic; and, if Φ is generic, then these complexes resolve half of the divisor class group of $R/I_\mathfrak{g}(\Phi)$. The family $\{\mathcal{C}^i_\Phi\}$ exhibits duality; and, if $-1\le i\le\mathfrak{f}-\mathfrak{g}+1$, then the complex $\mathcal{C}^i_\Phi$ exhibits depth-sensitivity with respect to the ideal $I_\mathfrak{g}(\Phi)$ in the sense that the tail of $\mathcal{C}^i_\Phi$ of length equal to grade$(I_\mathfrak{g}(\Phi))$ is acyclic. The entries in the differentials of $\mathcal{C}^i_\Phi$ are linear in the entries of Φ at every position except at one, where the entries of the differential are $\mathfrak{g}\times\mathfrak{g}$ minors of Φ.

This paper expands the family $\{\mathcal{C}^i_\Phi\}$ to a family of complexes $\{\mathcal{C}^{i,a}_\Phi\}$ for integers i and a with $1\le a\le\mathfrak{g}$. The entries in the differentials of $\{\mathcal{C}^{i,a}_\Phi\}$ are linear in the entries of Φ at every position except at two consecutive positions. At one of the exceptional positions the entries are $a\times a$ minors of Φ, at the other exceptional position the entries are $(\mathfrak{g}-a+1)\times(\mathfrak{g}-a+1)$ minors of Φ.

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[1] The author was partially supported by the Simons Foundation grant number 233597.

http://dx.doi.org/10.1016/j.jalgebra.2016.03.037

The complexes $\{\mathcal{C}_\Phi^i\}$ are equal to $\{\mathcal{C}_\Phi^{i,1}\}$ and $\{\mathcal{C}_\Phi^{i,\mathfrak{g}}\}$. The complexes $\{\mathcal{C}_\Phi^{i,a}\}$ exhibit all of the properties of $\{\mathcal{C}_\Phi^i\}$. In particular, if $-1 \le i \le \mathfrak{f}-\mathfrak{g}$ and $1 \le a \le \mathfrak{g}$, then $\mathcal{C}_\Phi^{i,a}$ exhibits depth-sensitivity with respect to the ideal $I_\mathfrak{g}(\Phi)$.

Contents

1. Introduction

Let R be a commutative Noetherian ring and F and G be free R-modules of rank $\mathfrak{f}$ and $\mathfrak{g}$, respectively, with $\mathfrak{g} \le \mathfrak{f}$. Recall that, for each R-module homomorphism

$$\Phi : G^* \to F \tag{1.0.1}$$

there is a family of generalized Eagon–Northcott complexes $\{\mathcal{C}_\Phi^i\}$. (See, for example, Definition 3.1, [13, Appendix A.2], [3, 2.16], or [20]. A more complete history of these complexes may be found in the comments on page 26 in [3].) If

$$-1 \le i \le \mathfrak{f}-\mathfrak{g}+1, \tag{1.0.2}$$

then $\mathcal{C}_\Phi^i$ has length $\mathfrak{f}-\mathfrak{g}+1$; and, if $\mathfrak{f}-\mathfrak{g}+1 \le \operatorname{grade} I_\mathfrak{g}(\Phi)$, then $\mathcal{C}_\Phi^i$ is acyclic for i satisfying (1.0.2). Furthermore, the complexes $\mathcal{C}_\Phi^i$, for i satisfying (1.0.2), exhibit depth-sensitivity. In particular, if $s \le \operatorname{grade} I_\mathfrak{g}(\Phi)$ for some integer s with $0 \le s \le f-g+1$, then $\mathrm{H}_j(\mathcal{C}_\Phi^i) = 0$ for $\mathfrak{f}-\mathfrak{g}+2-s \le j$ and i satisfying (1.0.2). In the generic situation, the complexes $\mathcal{C}_\Phi^i$, with i satisfying (1.0.2), resolve the Cohen–Macaulay elements of the divisor class group of the determinantal ring $R/I_\mathfrak{g}(\Phi)$. The complexes $\mathcal{C}_\Phi^i$, with i in the range (1.0.2), exhibit duality:

$$\mathcal{C}_\Phi^i \cong \text{a shift of } \operatorname{Hom}_R(\mathcal{C}_\Phi^j, R) \text{ in homological degree,}$$