Journal of Algebra 460 (2016) 102–127

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Rees algebras and almost linearly presented ideals

Jacob A. Boswell, Vivek Mukundan *

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

ARTICLE INFO

Article history:
Received 18 May 2015
Available online 4 May 2016
Communicated by Kazuhiko Kurano

Keywords:
Rees algebras
Defining ideals
Saturations
Iterated Jacobian duals
Almost linearly presented ideals
Second analytic deviation one ideals

ABSTRACT

Consider a grade 2 perfect ideal I in $R = k[x_1, \cdots, x_d]$ which is generated by forms of the same degree. Assume that the presentation matrix φ is almost linear, that is, all but the last column of φ consist of entries which are linear. For such ideals, we find explicit forms of the defining ideal of the Rees algebra $\mathcal{R}(I)$. We also introduce the notion of iterated Jacobian duals.

1. Introduction

In this paper we study the defining ideal of the Rees algebra of ideals in a commutative ring. The Rees algebra $\mathcal{R}(I)$ of an ideal I in a commutative ring R is defined to be $\mathcal{R}(I) = R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$. The defining ideal of the Rees algebra is the kernel $\mathcal{A}$ of an epimorphism $\Psi : R[T_1, \dots, T_m] \to \mathcal{R}(I)$ given by $\Psi(T_i) = \alpha_i t$, where $I = (\alpha_1, \dots, \alpha_m)$. Rees algebras provide an algebraic realization for the concept of blowing up a variety along a subvariety. The search for the implicit equations defining the Rees algebra is a classical and fundamental problem which has been studied for many decades. Some of the results in this direction include [29,11,23,17,13,6,20,7,4,3,19,5,21].

* Corresponding author.
E-mail addresses: jaboswel@purdue.edu (J.A. Boswell), vmukunda@purdue.edu (V. Mukundan).

http://dx.doi.org/10.1016/j.jalgebra.2016.03.035
0021-8693/

An important object in the study of Rees algebras is the symmetric algebra. The symmetric algebra $\mathrm{Sym}(I)$ of an ideal I has a presentation

$$\mathrm{Sym}(I) \cong R[T_1, \ldots, T_m]/\mathcal{L},$$

where $\mathcal{L} = ([T_1 \cdots T_m] \cdot \varphi)$ and φ is a presentation matrix of I. The map Ψ above factors through the symmetric algebra. So it is enough to study the kernel of the map $\mathrm{Sym}(I) \to \mathcal{R}(I)$. Traditionally, techniques for computing the defining ideal of $\mathcal{R}(I)$ often revolved around the notion of *Jacobian dual*. For a commutative ring R and an ideal I with a presentation $R^s \xrightarrow{\varphi} R^m \to I \to 0$, the *Jacobian dual* of φ is defined to be a matrix $B(\varphi)$ with linear entries in $R[T_1, \ldots, T_m]$ such that

$$[T_1 \cdots T_m] \cdot \varphi = [a_1 \cdots a_r] \cdot B(\varphi), \text{ where } I_1(\varphi) \subseteq (a_1, \ldots, a_r). \tag{1.1}$$

In the literature, the defining ideal of Rees algebras have been studied in great detail for many classes of ideals. For example, ideals generated by regular sequences (or d-sequences, [11]) and grade 2 perfect ideals with linear presentation ([23]). We restrict our study to the case where $R = k[x_1, \ldots, x_d]$ and $I = (\alpha_1, \ldots, \alpha_m)$ is a grade two perfect ideal minimally generated by homogeneous elements of the same degree. Using the Hilbert–Burch theorem, such an ideal can be realized as the ideal generated by the maximal minors of a $m \times m-1$ matrix with homogeneous entries of constant degree along each column. We further restrict the presentation matrix φ of I to be *almost linearly presented*, that is, all but the last column of φ are linear and the last column consists of homogeneous entries of arbitrary degree $n \geq 1$. The G_d condition is also an important ingredient in the study of $\mathcal{A}$. Here, the G_d condition means that $\mu(I_p) \leq \mathrm{ht}\ p$ for every $p \in V(I)$ with $\mathrm{ht}\ p \leq d-1$. An earlier study of Rees algebra of ideals of this type, when $d = 2$, was done by Kustin, Polini and Ulrich in [19]. We generalize their results for $d > 2$ and also present another form of the defining ideal of $\mathcal{R}(I)$.

The G_d condition forces some power of the ideal $(\underline{x})$ to annihilate the kernel of the map $\mathrm{Sym}(I) \to \mathcal{R}(I)$ i.e., $\mathcal{A} = \mathcal{L} : (\underline{x})^\infty$. Since $\dim \mathcal{R}(I) = d+1$, notice that $\mathcal{A} = \mathcal{L} : (\underline{x})^\infty$ is a prime ideal of height $m-1$.

One of the recurring features of the proofs in this paper is that the ideal $I_d(B(\varphi'))$ attains the maximum possible height, namely $m-d-1$ (where φ' is a matrix obtained from φ by removing the last column). This led us to study $\mathcal{L} : (\underline{x})^\infty$ in a more general setting.

Theorem 1.1. *Let R be a Cohen Macaulay local ring containing a field k and $\underline{a} = a_1, \ldots, a_r$ an R-regular sequence with $r > 0$. Let $S = R[T_1, \ldots, T_m]$ with $T_1, \ldots, T_m$ indeterminates over R and ψ be an $r \times s$ matrix with entries in $k[T_1, \ldots, T_m]$ so that each column consists of homogeneous elements of the same positive degree. If $(\underline{a} \cdot \psi) :_S (\underline{a})^\infty$*