Journal of Algebra 460 (2016) 143–180

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Species and non-commutative $\mathbb{P}^1$'s over non-algebraic bimodules ☆

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ARTICLE INFO

Article history:
Received 15 March 2016
Available online 1 June 2016
Communicated by Michel Van den Bergh

MSC:
primary 16G20
secondary 14A22

Keywords:
Representations of quivers
Non-commutative algebraic geometry
Non-commutative curves

ABSTRACT

We study non-commutative projective lines over not necessarily algebraic bimodules. In particular, we give a complete description of their categories of coherent sheaves and show they are derived equivalent to certain bimodule species. This allows us to classify modules over these species and thus generalize, and give a geometric interpretation for, results of C. Ringel [21].

Throughout this paper, K, K_0, and K_1 denote fields of characteristic $\neq 2$.

1. Introduction

The theory of tilting bundles has grown remarkably over the last few decades, connecting two otherwise unrelated fields of mathematics, algebraic geometry and representation

☆ The first author was supported by the Australian Research Council, Grant DP130100513.
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http://dx.doi.org/10.1016/j.jalgebra.2016.03.048
0021-8693/

theory. The simplest example occurs with the projective line $\mathbb{P}^1 = \mathbb{P}(K^2)$ which is derived equivalent to the Kronecker algebra or species $\Lambda = \begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$. Since both $\mathsf{Coh}\mathbb{P}^1$ and Λ are hereditary, this means that the indecomposable Λ-modules are in bijective correspondence with indecomposable sheaves on $\mathbb{P}^1$. More precisely, the indecomposable irregular modules correspond to the line bundles on $\mathbb{P}^1$ whilst the regular modules correspond to the torsion sheaves.

In the 1970s Dlab and Ringel [5,21] studied generalizations of the Kronecker algebra above where K^2 is replaced by a $K_0 - K_1$-bimodule V which is constant dimension two on each side (see Section 3). Corresponding bimodule generalizations $\mathbb{P}^{nc}(V)$ of the projective line appeared much later with the work of Patrick [18], Van den Bergh [23], and Nyman [16]. Here one defines a non-commutative version of the symmetric algebra $A = \mathbb{S}^{nc}(V)$, which is a $\mathbb{Z}$-indexed algebra closely related to Dlab and Ringel's preprojective algebra [6]. Using the non-commutative projective geometry of Artin–Zhang [3], we may then define $\mathbb{P}^{nc}(V) = \mathsf{Proj}\, A$ to be a certain quotient category of the category of graded A-modules (see Section 7). However, most research to date has concentrated on the case where the bimodule V is *algebraic* in the sense that there exists a common subfield k of K_0 and K_1 which acts centrally on V and K_i/k is finite. This hypothesis is usually included because it guarantees Hom-finiteness and thus Serre duality.

The goal of this paper is to study non-commutative projective lines and bimodule species without assuming the algebraic hypothesis. In particular, we will re-interpret and extend classic results of Ringel [21] by exploring derived equivalences between $\mathbb{P}^{nc}(V)$ and the corresponding bimodule species

$$\Lambda = \begin{pmatrix} K_0 & V \\ 0 & K_1 \end{pmatrix}.$$

It seems that the non-commutative projective line is the more amenable object to study and hence our point of view is that non-commutative projective geometry provides a profitable way to study and understand bimodule species. This is partly because many of the familiar notions of algebraic geometry (torsion, dimension, Hilbert functions) extend to sheaves on $\mathbb{P}^{nc}(V)$. For example, each unit $e_i \in A_{ii}$, $i \in \mathbb{Z}$ gives rise to a line bundle $\mathcal{A}_i$ which is the analogue of $\mathcal{O}(-i)$. We have the following generalization of Grothendieck's splitting theorem which recovers some results in [5] and [6], and generalizes [16, Theorem 3.14].

Theorem 1.1. *Any coherent sheaf on $\mathbb{P}^{nc}(V)$ is a direct sum of a torsion free sheaf with a torsion sheaf. Every torsion free sheaf is a direct sum of $\mathcal{A}_i$. In particular, the indecomposable irregular Λ-modules are induced by multiplication in the non-commutative symmetric algebra and are of the form*

$$A_{i0} \otimes_K V \to A_{i1} \quad or \quad A^*_{0i-2} \otimes_K V \to A^*_{1i-2}.$$

It is in the study of regular Λ-modules or equivalently, torsion sheaves on $\mathbb{P}^{nc}(V)$, that non-commutative projective geometry seems to have something really new to contribute,