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# Total positivity, Schubert positivity, and geometric Satake

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ABSTRACT

Let $G$ be a simple, simply-connected complex algebraic group, and let $X \subset G^\vee$ be the centralizer of a principal nilpotent. Ginzburg and Peterson independently related the ring of functions on $X$ with the homology ring of the affine Grassmannian $\mathrm{Gr}_G$. Peterson furthermore connected $X$ to the quantum cohomology rings of partial flag varieties $G/P$.
In this paper we study three notions of positivity for $X$: (1) *Schubert positivity* arising via Peterson's work, (2) Lusztig's *total positivity* and (3) *Mirković–Vilonen positivity* obtained from the MV-cycles in $\mathrm{Gr}_G$. The first main theorem establishes that these three notions of positivity coincide. Our second main theorem proves a parametrization of the totally nonnegative part of $X$, confirming a conjecture of the second author.
In type A the parametrization and relationship with Schubert positivity were proved earlier by the second author. Here we tackle the general type case and also introduce a crucial new connection with the affine Grassmannian and geometric Satake correspondence.


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## 1. Introduction

Let $G$ be a simply connected, semisimple complex linear algebraic group, split over $\mathbb{R}$, and let $G^\vee$ be its Langlands dual group (over $\mathbb{C}$). The Peterson variety $\mathcal{Y}$ may be viewed as the compactification of the stabilizer $X := G^\vee_F$ of a standard principal nilpotent $F$ in $(\mathfrak{g}^\vee)^*$ (with respect to the coadjoint representation of $G^\vee$), which one obtains by embedding $X$ into the Langlands dual flag variety $G^\vee/B^\vee_-$ and taking the closure there.

Ginzburg [11] and Peterson [31] independently showed that the coordinate ring $\mathcal{O}(X)$ of the variety $X$ was isomorphic to the homology $H_*(\mathrm{Gr}_G)$ of the affine Grassmannian $\mathrm{Gr}_G$ of $G$, and Peterson discovered moreover that the compactification $\mathcal{Y}$ encodes the quantum cohomology rings of all of the flag varieties $G/P$. Peterson's remarkable work in particular exhibited explicit homomorphisms between localizations of $qH^*(G/P, \mathbb{C})$ and $H_*(\mathrm{Gr}_G, \mathbb{C})$ taking quantum Schubert classes $\sigma^P_w$ to affine homology Schubert classes $\xi_x$. These homomorphisms were verified in [24].

The first aim of this paper is to compare different notions of positivity for the real points of $X$: (i) the *affine Schubert positive* part $X^{\mathrm{af}}_{>0}$ where affine Schubert classes $\xi_x$ take positive values via Ginzburg and Peterson's isomorphism $H_*(\mathrm{Gr}_G) \simeq \mathcal{O}(X)$; (ii) the *totally positive* part $X_{>0} := X \cap U^\vee_{-,>0}$ in the sense of Lusztig [26]; and (iii) the *Mirković–Vilonen positive* part $X^{\mathrm{MV}}_{>0}$ where the classes of the Mirković–Vilonen cycles from the geometric Satake correspondence [29] take positive values.

Our first main theorem (Theorem 7.1) states that these three notions of positivity coincide. For $G$ of type $A$ the coincidence $X^{\mathrm{af}}_{>0} = X_{>0}$ was already established in [35], where instead of $X^{\mathrm{af}}_{>0}$, the notion of *quantum Schubert positivity* was used. In general quantum Schubert positivity is possibly weaker than affine Schubert positivity. It follows from [35] that the notions coincide in type A, and we verify that they coincide in type C in Appendix A.

The notion of Mirković–Vilonen positivity does not appear to have been studied in the literature before. We note that the MV-basis is expected to coincide with Lusztig's semicanonical basis which is distinct from the canonical basis used in Lusztig's approach to total positivity. So the coincidence $X^{MV}_{>0} = X_{>0}$ might not be immediately expected. As for the comparison $X^{MV}_{>0} = X^{\mathrm{af}}_{>0}$ we note that the classes of MV-cycles span $H_*(\mathrm{Gr}_G)$ over $\mathbb{C}$, but the $\mathbb{Z}$-lattice spanned by MV-cycles is known to be strictly contained in the lattice spanned by the Schubert basis.

Our second main theorem (Theorem 7.3) is a parametrization of the totally positive $X_{>0}$ and totally nonnegative $X_{\geq 0}$ parts of $X$. We show that they are homeomorphic to $\mathbb{R}^n_{>0}$ and $\mathbb{R}^n_{\geq 0}$ respectively. This was conjectured by the second author in [35] where it was established in type $A$. In type $A_n$ we have that $X = G^\vee_F$ is the $n$-dimensional subgroup of lower-triangular unipotent Toeplitz matrices, and thus the parametrization $X_{\geq 0} \simeq \mathbb{R}^n_{\geq 0}$ is a "finite-dimensional" analogue of the Edrei–Thoma theorem [7] parametrizing *infinite* totally nonnegative Toeplitz matrices, appearing in the classification of the characters of the infinite symmetric group. The results of this article give an arbitrary type generalization.