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Properly integral polynomials over the ring of integer-valued polynomials on a matrix ring

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ABSTRACT

Let D be a domain with fraction field K, and let $M_n(D)$ be the ring of $n \times n$ matrices with entries in D. The ring of integer-valued polynomials on the matrix ring $M_n(D)$, denoted $\mathrm{Int}_K(M_n(D))$, consists of those polynomials in $K[x]$ that map matrices in $M_n(D)$ back to $M_n(D)$ under evaluation. It has been known for some time that $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ is not integrally closed. However, it was only recently that an example of a polynomial in the integral closure of $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ but not in the ring itself appeared in the literature, and the published example is specific to the case $n = 2$. In this paper, we give a construction that produces polynomials that are integral over $\mathrm{Int}_K(M_n(D))$ but are not in the ring itself, where D is a Dedekind domain with finite residue fields and $n \geq 2$ is arbitrary. We also show how our general example is related to P-sequences for $\mathrm{Int}_K(M_n(D))$ and its integral closure in the case where D is a discrete valuation ring.

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1. Introduction

When D is a domain with field of fractions K, the ring of integer-valued polynomials on D is $\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$. Such rings have been extensively studied over the past several decades; the reader is referred to [5] for standard results on these objects. More recently, attention has turned to the consideration of integer-valued polynomials on algebras [6,8–12,17,18,20–23]. The typical approach for this construction is to take a torsion-free D-algebra A that is finitely generated as a D-module and such that $A \cap K = D$. Then, we define $\mathrm{Int}_K(A)$ to be the set of polynomials in $K[x]$ that map elements of A back to A under evaluation. That is, $\mathrm{Int}_K(A) := \{f \in K[x] \mid f(A) \subseteq A\}$, which is a subring of $\mathrm{Int}(D)$. (Technically, evaluation of $f \in K[x]$ at elements of A is performed in the tensor product $K \otimes_D A$ by associating K and A with their canonical images $K \otimes 1$ and $1 \otimes A$. In practice, however, it is usually clear how to perform the evaluation without the formality of tensor products.)

Depending on the choice of A, the ring $\mathrm{Int}_K(A)$ can exhibit similarities to, or stark differences from, $\mathrm{Int}(D)$. For instance, if A is the ring of integers of a number field (viewed as a $\mathbb{Z}$-algebra), then $\mathrm{Int}_{\mathbb{Q}}(A)$ is—like $\mathrm{Int}(\mathbb{Z})$—a Prüfer domain [17, Thm. 3.7], hence is integrally closed. In contrast, when $A = M_n(\mathbb{Z})$ is the algebra of $n \times n$ matrices with entries in $\mathbb{Z}$, $\mathrm{Int}_{\mathbb{Q}}(A)$ is not integrally closed (although its integral closure is a Prüfer domain) [17, Sec. 4]. In a more general setting, it is known [5, Thm. VI.1.7] that if D is a Dedekind domain with finite residue fields, then $\mathrm{Int}(D)$ is a Prüfer domain, and so is integrally closed. The motivation for this paper was to show, by giving a form for a general counterexample, that $\mathrm{Int}_K(M_n(D))$ is not integrally closed. In this vein, we make the following definition.

Definition 1.1. A polynomial $f \in K[x]$ will be called *properly integral* over $\mathrm{Int}_K(A)$ if f lies in the integral closure of $\mathrm{Int}_K(A)$, but $f \notin \mathrm{Int}_K(A)$.

Note that the integral closure of $\mathrm{Int}_K(A)$ in its field of fractions $K(x)$ is contained in $K[x]$, so that $\mathrm{Int}_K(A)$ is integrally closed if and only if there are no properly integral polynomials over $\mathrm{Int}_K(A)$. It has been known for some time that $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ is not integrally closed. However, the first published example of a properly integral polynomial over $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ was given only recently by Evrard and Johnson in [9], and only for the case $n = 2$. We will give a general construction for a properly integral polynomial over $\mathrm{Int}_K(M_n(D))$, where D is a Dedekind domain with finite residue rings, and $n \geq 2$ is arbitrary.

The theorems in this paper can be seen as complementary to the work of Evrard and Johnson. Their results relied heavily on the P-orderings and P-sequences of Bhargava [4] and the generalizations of these in [15]. In the case where $D = \mathbb{Z}$, a properly integral polynomial $f(x) = g(x)/p^k$ (where $g \in \mathbb{Z}[x]$, p^k is a prime power, and p does not divide g) over $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ produced by using the methods and p-sequences in [9] is optimal in the sense that f has minimal degree among all properly integral polynomials of the form