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Two remarks about non-vanishing elements in finite groups

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ABSTRACT

Let G be a finite group. We prove that for $x \in G$ we have $\chi(x) \neq 0$ for all irreducible characters χ of G iff the class sum of x in the group algebra over $\mathbb{C}$ is a unit. From this we conclude that if G has a normal p-subgroup V and a Hall p'-subgroup, then G has non-vanishing elements different from 1. Hence we get another proof that a finite solvable group always has non-trivial non-vanishing elements. Moreover, we give an example for a finite solvable group G which has a non-vanishing involution not contained in an abelian normal subgroup of G.

Let G be a finite group. In [2] the authors introduced the notion of a non-vanishing element of G. An element $x \in G$ is called *non-vanishing* in G if $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$.

If N is a normal subgroup of G and x is non-vanishing in G, then of course xN is non-vanishing in G/N. If however H is a subgroup of G and $x \in H$, then neither x non-vanishing in G implies x non-vanishing in H nor does the other implication hold. In this paper, we prove that an element is non-vanishing if and only if the class sum is a unit in the group algebra over $\mathbb{C}$. Therefore, for $x \in H \leq G$ with $x^G = x^H$, we get that

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x is non-vanishing in H iff x is non-vanishing in G, so both implications in fact hold in this situation. As a corollary, we get a variation of a result of Miyamoto (see [3]) and another proof that a solvable group always contains non-trivial non-vanishing elements.

In [2] the authors proved that if G is solvable and $x \in G$ is non-vanishing, then $xF(G)$ is a 2-element, where $F(G)$ denotes the Fitting subgroup of G. They conjecture that in fact $x \in F(G)$ must hold. They prove that if this conjecture fails, then there is a solvable group G and a non-vanishing involution in G which is not contained in an abelian normal subgroup ([2, Theorem (4.4)]), and remark that they were unable to construct an example where this happens. In the second part of this paper we give an example for this situation.

Parts of this paper have been taken from the author's diploma thesis. The author likes to thank the referee for her or his valuable comments.

1. Non-vanishing elements

In the following G is always a finite group. For $S \subseteq G$ let $\widehat{S} = \sum_{g \in S} g \in \mathbb{C}G$. For $\chi \in \mathrm{Irr}(G)$ let $e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g$ be the central idempotent of $\mathbb{C}G$ corresponding to χ, and $\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}$ the algebra homomorphism defined by $\omega_\chi(\widehat{g^G}) = \frac{|g^G|\chi(g)}{\chi(1)}$ (see [1, (2.12) and the remark before (3.7)]). Recall that we have $1 = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi$, $e_\chi e_\psi = \delta_{\chi,\psi} e_\chi$, $\omega_\chi(e_\psi) = \delta_{\chi,\psi}$ for all $\chi, \psi \in \mathrm{Irr}(G)$ and $z = \sum_{\chi \mathrm{Irr}(G)} \omega_\chi(z) e_\chi$ for all $z \in Z(\mathbb{C}G)$.

Definition 1.1. An element $x \in G$ is called *non-vanishing* in G if $\chi(x) = 0$ for all $\chi \in \mathrm{Irr}(G)$.

Proposition 1.2. *Let $x \in G$. The following are equivalent:*

1. *x is non-vanishing in G.*
2. *The class sum $\widehat{x^G} = \sum_{g \in x^G} g$ is a unit of $\mathbb{C}G$.*

Proof. We have $z = \sum_{\chi \in \mathrm{Irr}(G)} \omega_\chi(z) e_\chi$ for every $z \in Z(\mathbb{C}G)$, hence z is a unit of $\mathbb{C}G$ iff $\omega_\chi(z) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$. In this case we have $z^{-1} = \sum_{\chi \in \mathrm{Irr}(G)} \omega_\chi(z)^{-1} e_\chi$. For $x \in G$ and $\chi \in \mathrm{Irr}(G)$ we have $\omega_\chi(\widehat{x^G}) = \frac{|x^G|\chi(x)}{\chi(1)}$, thus $\omega_\chi(\widehat{x^G}) \neq 0$ iff $\chi(x) \neq 0$. Thus the claim follows. $\square$

Corollary 1.3. *Let $x \in H \leq G$ such that $G = C_G(x)H$. Then x is non-vanishing in G if and only if x is non-vanishing in H.*

Proof. Since $\dim_{\mathbb{C}} \mathbb{C}H$ is finite, an element $z \in \mathbb{C}H$ is invertible in $\mathbb{C}H$ if and only if z is not a zero divisor. Hence if z is invertible in $\mathbb{C}G$, then z is already invertible in $\mathbb{C}H$. By assumption $x^H = x^G$ and hence $\widehat{x^H} = \widehat{x^G}$. Thus the claim follows. $\square$