Journal of Algebra 460 (2016) 380–386

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Generating minimally transitive permutation groups

Gareth M. Tracey

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom

A R T I C L E   I N F O

Article history:
Received 13 June 2015
Available online 1 June 2016
Communicated by Martin Liebeck

Keywords:
Group theory
Finite permutation groups
Minimal generation of finite groups
Crown-based powers
Finite simple groups

A B S T R A C T

We prove that each minimally transitive permutation group of degree n can be generated by $\mu(n)+1$ elements, where $\mu(n) := \max\{m : \text{there exists a prime } p \text{ such that } p^m \text{ divides } n\}$.

1. Introduction

A transitive permutation group $G \leq S_n$ is called *minimally transitive* if every proper subgroup of G is intransitive. In this paper, we consider the minimal number of elements $d(G)$ required to generate such a group G, in terms of its degree n. For the prime factorisation $n = \prod_{p \text{ prime}} p^{n(p)}$ of n, we will write $\omega(n) := \sum_p n(p)$ and $\mu(n) := \max\{n(p) : p \text{ prime}\}$.

The question of bounding $d(G)$ in terms of n was first considered by Shepperd and Wiegold in [13]; there, they prove that every minimally transitive group of degree n can

E-mail address: G.M.Tracey@warwick.ac.uk.

http://dx.doi.org/10.1016/j.jalgebra.2016.04.021
0021-8693/

be generated by $\omega(n)$ elements. It was then suggested by Pyber (see [12]) to investigate whether or not $\mu(n)+1$ elements would always suffice. A. Lucchini gave a partial answer to this question in [9], proving that: *if G is a minimally transitive group of degree n, and $\mu(n)+1$ elements are not sufficient to generate G, then $\omega(n) \geq 2$ and $d(G) \leq \lfloor \log_2(\omega(n)-1)+3 \rfloor$.*

In this note, we offer a complete solution to the problem, proving

Theorem 1.1. *Let G be a minimally transitive permutation group of degree n. Then $d(G) \leq \mu(n)+1$.*

Our approach follows along the same lines as Lucchini's proof of the main theorem in [9]. Indeed, his methods suffice to prove Theorem 1.1 in the case when a minimal normal subgroup of G is abelian. Thus, our main efforts will be concerned with the case when a minimal normal subgroup of G is a direct product of isomorphic nonabelian simple groups. The key step in this direction is Lemma 3.1, which we prove in Section 3. We use Section 2 to outline the method of *crown-based powers* due to F. Dalla Volta and Lucchini; this will serve as the basis for our arguments. Finally, we prove Theorem 1.1 in Section 4.

2. Crown-based powers

In this section, we outline an approach to study the question of finding the minimal number of elements required to generate a finite group, which is due to F. Dalla Volta and A. Lucchini. So let G be a finite group, with $d(G)=d>2$, and let M be a normal subgroup of G, maximal with the property that $d(G/M)=d$. Then G/M needs more generators than any proper quotient of G/M, and hence, as we shall see below, G/M takes on a very particular structure.

We describe this structure as follows: let L be a finite group, with a unique minimal normal subgroup N. If N is abelian, then assume further that N is complemented in L. Now, for a positive integer k, set L_k to be the subgroup of the direct product L^k defined as follows

$$L_k := \{(x_1, x_2, \ldots, x_k) : x_i \in L,\, Nx_i = Nx_j \text{ for all } i, j\}$$

Equivalently, $L_k := \mathrm{diag}(L^k)N^k$, where $\mathrm{diag}\,(L^k)$ denotes the diagonal subgroup of L^k. The group L_k is called the *crown-based power of L of size k.*

We can now state the theorem of Dalla Volta and Lucchini.

Theorem 2.1 *([2], Theorem 1.4). Let G be a finite group, with $d(G) \geq 3$, which requires more generators than any of its proper quotients. Then there exists a finite group L, with a unique minimal normal subgroup N, which is either nonabelian or complemented in L, and a positive integer $k \geq 2$, such that $G \cong L_k$.*