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Non-vanishing elements in finite groups

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ABSTRACT

Many results have been established about determining whether or not an element evaluates to zero on an irreducible character of a group. In this note it is shown that if a group G has a normal p-subgroup N, and $x \in N$ lies in the centre of a Sylow p-subgroup of G, then no irreducible character of G vanishes on x.

Let G be a finite group and $\chi \in \mathrm{Irr}(G)$, an irreducible character of G. A classical result of Burnside says if χ is non-linear, that is $\chi(1) \neq 1$, then there is at least one element g in G such that $\chi(g) = 0$. If one considers conjugacy classes, a natural dual to irreducible characters, then g being a central element in G implies that $|\chi(g)| = \chi(1)$ and thus g does not evaluate to zero on any irreducible character. However, a non-central element g may also not evaluate to zero on any irreducible character, for example the 3-cycles in $\mathrm{Sym}(3)$. Elements which do not evaluate to zero on any irreducible character of a group are called non-vanishing. The study of non-vanishing elements was first introduced in [4], where the authors showed for soluble groups any non-vanishing element g in a group G must reduce to a 2-element in $G/F(G)$. In [2] this result was generalised to any group,

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in particular it was shown that if an element g is non-vanishing in G and the order of g is coprime to 6, then g lies in $F(G)$.

We note that there has been a recent interest in the literature asking about how much group structure is determined by the vanishing conjugacy class sizes. In particular, in [3] and [1], the authors have generalised arithmetical results upon conjugacy classes to vanishing conjugacy classes. Thus the determination of non-vanishing elements would provide further machinery for this recent topic of research.

The aim of this note is to generalise a result appearing in [4], that is [4, Theorem A], which says if a group has a normal Sylow p-subgroup P, then all the elements in $Z(P)$ are non-vanishing. A variant of this result was considered in [8], where the author showed that if a group has a normal elementary abelian p-subgroup A and P is a Sylow p-subgroup, then the elements in $Z(P) \cap A$ are non-vanishing. In particular, we first show that the result of [8] holds if A is a normal abelian p-subgroup. From this we deduce the result holds if A is a normal p-subgroup. Note that from this new result, [4, Theorem A] follows by setting $A = P$.

Theorem. *Let G be a finite group which contains a non-trivial normal p-subgroup N for p a prime. Then any $x \in N$ such that p does not divide $|G : C_G(x)|$ is non-vanishing in G.*

First we give the following preliminary result which considers when a sum of roots of unity is equal to zero.

Lemma. *Let $\Xi := \{\xi_1, \dots, \xi_t\}$ be a set of p^n-th roots of unity, for some number $n \geq 1$, such that $\xi_1 + \dots + \xi_t = 0$. Then the sum can be split into sums of the form $\xi + \xi^{p^{a-1}+1} + \dots + \xi^{(p-1)p^{a-1}+1}$, for possibly various numbers $1 \leq a \leq n$, where $\xi^{kp^{a-1}+1} \in \Xi$ for $0 \leq k \leq p-1$ and each such subsum equals zero.*

Proof. Let ξ be an element in $\{\xi_i \mid 1 \leq i \leq t\}$ of maximal order, i.e. ξ is a primitive p^a-th root of unity and $\xi_i^{p^a} = 1$ for all i. It is enough to prove that $\xi, \xi^{p^{a-1}+1}, \dots, \xi^{(p-1)p^{a-1}+1} \in \Xi$, as then

$$\xi + \xi^{p^{a-1}+1} + \dots + \xi^{(p-1)p^{a-1}+1} = \xi(1 + \xi^{p^{a-1}} + \dots + \xi^{(p-1)p^{a-1}}) = 0$$

and inductively from $\Xi \backslash \{\xi, \xi^{p^{a-1}+1}, \dots \xi^{(p-1)p^{a-1}+1}\}$ repeat the argument.

Assume $\xi_1 = \xi$, which is a primitive p^a-th root of unity, so that each ξ_i is a power of ξ. Pick r minimal such that $\Sigma_{j=1}^r \xi^{b_j} = 0$ with $\xi^{b_j} \in \Xi$, where $b_i \leq b_{i+1} \leq p^a$ and $b_1 = 1$. Then it follows that ξ is a root to the polynomial $\Sigma_{j=1}^r X^{b_j}$. As $\Phi_{p^a}(X) = 1 + X^{p^{a-1}} + \dots + X^{(p-1)p^{a-1}}$ is the minimal polynomial for ξ it follows that

$$\Sigma_{j=1}^r X^{b_j} = \Phi_{p^a}(X) g(X),$$

for some polynomial $g \in \mathbb{Z}[X]$.