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Arithmetic and geometry of the Hecke groups

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A B S T R A C T

We study the arithmetic and geometry properties of the Hecke group G_q. In particular, we prove that G_q has a subgroup X of index d, genus g with v_∞ cusps, and τ_2 (resp. v_{r_i}) conjugacy classes of elliptic elements that are conjugates of S (resp. R^{q/r_i}) if and only if (i) $2g-2+\tau_2/2+\sum_{i=1}^k v_{r_i}(1-1/r_i)+v_\infty = d(1/2-1/q)$, and (ii) $m_0 = 4g-4+\tau_2+2v_\infty+\sum_{i=1}^k v_{r_i}(2-q/r_i) \geq 0$ is a multiple of $q-2$. Note that if q is odd (resp. prime), then $m_0/(q-2) \in \mathbb{Z}$ (resp. $\mathbb{N} \cup \{0\}$) is a consequence of (i).

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1. Introduction

1.1. Let $q \geq 3$ be an integer. The (inhomogeneous) Hecke group (see [1]) G_q is defined to be the maximal discrete subgroup of $PSL(2,\mathbb{R})$ generated by S and T, where $\lambda_q = 2\cos(\pi/q)$,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}. \tag{1.1}$$

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Let $R = ST^{-1}$. Then R has order q and $\{S, R\}$ is a set of independent generators of G_q. Equivalently, G_q is a free product of $\langle S\rangle$ and $\langle R\rangle$. The main purpose of this article is to study the geometric and arithmetic properties of subgroups of finite index of G_q.

1.2. The set of cusps of G_q is $\mathbb{Q}[\lambda_q] \cup \{\infty\}$ if and only if $q = 3, 5$. We will give an inductive procedure (induction on the depth of q-gons) that enables us to generate the set of cusps of G_q (Lemma 3.2). As the index of G_q in $PSL(2, \mathbb{Z}[\lambda_q])$ is infinite if $q \geq 4$, it is important to characterise members of G_q. A simple algorithm that determines whether a matrix of $PSL(2, \mathbb{Z}[\lambda_q])$ belongs to G_q can be found in Proposition 3.7. It is our duty to point out that Proposition 3.7 is just an easy consequence of Rosen's study of λ_q continued fraction expansion of real numbers (see [10]) and that members of G_q are completely known only if $q = 4$ or 6 (see [3,12] or Remark 3.9 of the current article).

1.3. A set of generators $\{x_i\}$ of X is called a set of *independent generators* if X is a free product of the cyclic groups $\langle x_i\rangle$. G_q is a free product of $\langle S\rangle$ and $\langle R\rangle$. By Kurosh's Theorem, every subgroup X of finite index of G_q has a set of independent generators. Proposition 4.4 and Theorem 5.2 demonstrate how arithmetic and geometry can be combined to give an inductive procedure for finding a special polygon (fundamental domain) M_X and a set of independent generators I_X for X (the case q is a prime has been done in [6]). In particular, this is applied to the principal congruence subgroup of level 2, the commutator subgroup G_q' and subgroups of index 2 (subsection 5.4).

1.4. As a special case of the Hurwitz–Nielsen realisation problem, Millington [9] showed that as long as $d = 3\tau_2 + 4v_3 + 12g + 6t - 12$, then the modular group G_3 possesses a subgroup X of index d, such that $X \setminus \mathbb{H}$ has τ_2 (resp. v_3) elliptic points of order 2 (resp. 3), t cusps, and genus g. We are able to generalise this result to G_q by studying the Hecke–Farey symbols (see Section 6). Another application of our study of the special polygons is an easy method (calculation free) that determines the permutation representation of G_q on G_q/X. As a consequence, whether X is normal can be determined easily (see subsection 7.3). In the case $q = 3$, whether X is congruence can be determined as well (see subsection 7.4).

2. Tessellation of the upper half plane

Let D^* denote the $(2, q, \infty)$ triangle with vertices i, $e^{\pi i/q}$ and ∞. D^* is a fundamental domain of the Coxeter group G_q^* generated by reflections along the sides of D^*. Hecke group G_q is the subgroup of index 2 consists of all the orientation preserving isometries.

Let $\mathbb{H}$ be the union of the upper half plane and the set $\{g(\infty) : g \in G_q^*\}$. The G_q^* translates of D^* form a tessellation $\mathcal{I}^*$ of $\mathbb{H}$ (endowed with the hyperbolic metric) by $(2, q, \infty)$ triangles. The G_q^* translates of i, $e^{\pi i/q}$ and ∞ are called *even vertices, odd vertices* and *cusps* (*free vertices*) of $\mathcal{I}^*$ respectively. The G_q^* translates of the hyperbolic line joining i to ∞ (resp. $e^{\pi i/q}$ to ∞) are called *even edges* (resp. *odd edges*) of $\mathcal{I}^*$. The G_q^*

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