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On the first cohomology of automorphism groups of graph groups

Javier Aramayona *, Conchita Martínez-Pérez

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ABSTRACT

We study the (virtual) indicability of the automorphism group $\mathrm{Aut}(A_\Gamma)$ of the right-angled Artin group A_Γ associated to a simplicial graph Γ. First, we identify two conditions – denoted (B1) and (B2) – on Γ which together imply that $H^1(G,\mathbb{Z}) = 0$ for certain finite-index subgroups $G < \mathrm{Aut}(A_\Gamma)$. On the other hand we will show that (B2) is equivalent to the matrix group $\mathcal{H} = \mathrm{Im}(\mathrm{Aut}(A_\Gamma) \to \mathrm{Aut}(H_1(A_\Gamma))) < \mathrm{GL}(n,\mathbb{Z})$ not being virtually indicable, and also to $\mathcal{H}$ having Kazhdan's property (T). As a consequence, $\mathrm{Aut}(A_\Gamma)$ virtually surjects onto $\mathbb{Z}$ whenever Γ does not satisfy (B2). In addition, we give an extra property of Γ ensuring that $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$ virtually surject onto $\mathbb{Z}$. Finally, in the appendix we offer some remarks on the linearity problem for $\mathrm{Aut}(A_\Gamma)$.

1. Introduction

Automorphism groups of right-angled Artin groups (or *graph groups*, or *partially commutative groups*) form an interesting class of groups, as they "interpolate" between the two extremal cases of $\mathrm{Aut}(F)$, the automorphism group of a non-abelian free group, and the general linear group $\mathrm{GL}(n,\mathbb{Z})$.

* Corresponding author.
E-mail address: aramayona@gmail.com (J. Aramayona).

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In this paper we study the (non-)triviality of the first cohomology group of $\mathrm{Aut}(A_\Gamma)$ and certain classes of its finite-index subgroups; here, A_Γ denotes the right-angled Artin group defined by the simplicial graph Γ. Recall that a discrete group G is said to be *virtually indicable* if there exists a subgroup $G_0 < G$ of finite index with non-trivial first cohomology group; equivalently, G_0 admits a surjection onto $\mathbb{Z}$. We say that a compactly generated group G has *Kazhdan's property (T)* if every unitary representation of G that has almost invariant vectors has an invariant unit vector. Groups with *Kazhdan's property (T)* are not virtually indicable, however the converse is not true; for instance, $\mathrm{Aut}(F_3)$ has finite abelianization but does not enjoy property (T) [19,15,3].

As is often the case with properties of (automorphisms of) right-angled Artin groups, whether $H^1(\mathrm{Aut}(A_\Gamma), \mathbb{Z})$ vanishes or not depends on the structure of the underlying graph Γ. Below, we will identify a number of conditions on Γ ensuring that $\mathrm{Aut}(A_\Gamma)$ has (non-)trivial first cohomology. These conditions are phrased on the usual partial ordering of the vertex set $V(\Gamma)$ of Γ. Namely, given vertices $v, w \in V(\Gamma)$, we say that $v \leq w$ if $\mathrm{lk}(v) \subset \mathrm{st}(w)$; see section 2 for an expanded definition. We write $v \sim w$ to mean $v \leq w$ and $w \leq v$.

1.1. Finite abelianization

We first consider a property of a graph which guarantees that the partial ordering $\leq$ is "sufficiently rich". More concretely, we say that a simplicial graph Γ has property (B) if the following two conditions hold:

(B1) For all $u, v \in V(\Gamma)$ that are not adjacent, we have $u \sim v$;
(B2) For all $v, w \in V(\Gamma)$ with $v \leq w$, there exists $u \in V(\Gamma)$ such that $u \neq v, w$ and $v \leq u \leq w$.

We will prove that if Γ has property (B) then a natural class of finite-index subgroups of $\mathrm{Aut}(A_\Gamma)$ have finite abelianization. Before we state our first result, recall that the Torelli group $\mathcal{I}A_\Gamma$ is the kernel of the natural homomorphism $\mathrm{Aut}(A_\Gamma) \to \mathrm{Aut}(H_1(A_\Gamma)) = \mathrm{GL}(n, \mathbb{Z})$ where n denotes the number of vertices of Γ. In particular, we may see $\mathrm{Aut}(A_\Gamma)/\mathcal{I}A_\Gamma$ as a subgroup of $\mathrm{GL}(n, \mathbb{Z})$.

Our first result is:

Theorem 1.1. *Let Γ be a simplicial graph with property (B). If $G < \mathrm{Aut}(A_\Gamma)$ is a finite-index subgroup containing $\mathcal{I}A_\Gamma$, then $H^1(G, \mathbb{Z}) = 0$.*

Remark 1.2. Denote by F_k the free group on k letters. As we will see in Lemma 3.1 below, Γ has property (B) if and only if $A_\Gamma \cong F_{n_1} \times \ldots \times F_{n_k} \times \mathbb{Z}^a$, where $n_1, \ldots, n_k > 2$ and $a \neq 2$. Observe, however, that if w is a vertex corresponding to the abelian factor then for any other vertex v there is a transvection $t_{vw} \in \mathrm{Aut}(A_\Gamma)$ mapping $v \mapsto vw$ and fixing the rest of the generators. In particular, $\mathrm{Aut}(A_\Gamma)$ does not keep the factors invariant in