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Categorification of Virasoro–Magri Poisson vertex algebra

Seok-Jin Kang [a,1], Uhi Rinn Suh [b,*,2]

[a] *Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, 599 Gwanak-Ro, Seoul 151-747, Republic of Korea*
[b] *Research Institute of Mathematics, Seoul National University, 599 Gwanak-Ro, Gwanak-Gu, Seoul 151-747, Republic of Korea*

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ABSTRACT

Let Σ be the direct sum of algebra of symmetric groups $\mathbb{C}\Sigma_n$, $n \in \mathbb{Z}_{\geq 0}$. We show that the Grothendieck group $K_0(\Sigma)$ of the category of finite dimensional modules of Σ is isomorphic to the differential algebra of polynomials $\mathbb{Z}[\partial^n x \,|\, n \in \mathbb{Z}_{\geq}0]$. Moreover, we define m-th products $(m \in \mathbb{Z}_{\geq 0})$ on $K_0(\Sigma)$ which make the algebra $K_0(\Sigma)$ isomorphic to an integral form of the Virasoro–Magri Poisson vertex algebra. Also, we investigate relations between $K_0(\Sigma)$ and $K_0(N)$ where $K_0(N)$ is the direct sum of Grothendieck groups $K_0(N_n)$, $n \geq 0$, of finitely generated projective N_n-modules. Here N_n is the nil-Coxeter algebra generated by $n-1$ elements.

* Corresponding author.
E-mail addresses: sjkang@math.snu.ac.kr (S.-J. Kang), uhrisu1@math.snu.ac.kr (U.R. Suh).
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Introduction

A *Poisson vertex algebra (PVA)* arises in mathematical physics as the underlying algebraic structure of the classical field theory. Also it is connected to other algebraic structures in mathematical physics. For example, it is a quasi-classical limit of a family of vertex algebras, which can be seen as the algebraic structure appearing in the 2-dimensional conformal field theory. Moreover, PVAs with Hamiltonian operators are chiralizations of Poisson algebras which are related to the classical mechanics. Here the chiralization implies a Poisson algebra of finite algebraic dimension is connected to a PVA of infinite algebraic dimension. (See [5].)

In [9], Zhu constructed the maps called the Zhu maps which relate an associative algebra $Zhu_H(V)$ to a vertex algebra V with a Hamiltonian operator H. He proved that there is a one-to-one correspondence between irreducible positive energy modules over a vertex algebra V and irreducible modules over the Zhu algebra $Zhu_H(V)$. Analogously, De Sole and Kac [3] took the quasi-classical limit of the Zhu map and obtained the (classical) Zhu map which associates a Poisson algebra $Zhu_H(\mathcal{V})$ to a PVA $\mathcal{V}$ with a Hamiltonian operator H. Hence, from the point of view of representation theory, the Zhu map is a reasonable finitization map from a PVA to a Poisson algebra. Thus we consider a PVA as a chiralization of its Zhu algebra. The simplest example of Zhu algebras is $Zhu_H(\mathcal{V}) = \mathbb{C}[x]$ with the trivial Poisson bracket, where $\mathcal{V}$ is the Virasoro–Magri PVA and $H = L_0$ for an energy momentum field $L \in \mathcal{V}$.

On the other hand, Khovanov [6] showed that the direct sum of Grothendieck groups of finitely generated projective modules of the nil-Coxeter algebras is isomorphic to the polynomial algebra $\mathbb{Z}[x]$. More precisely, let N_n, $n \in \mathbb{Z}_{\geq 0}$, be the nil-Coxeter algebra generated by $n-1$ elements and let $K_0(N_n)$ be the Grothendieck group of the category N_n-pmod of finitely generated projective modules over N_n. Then $K_0(N) = \bigoplus_{n\geq 0} K_0(N_n)$ is isomorphic to $\mathbb{Z}[x]$. Moreover, he showed that the induction and restriction functors on the direct sum $\bigoplus_{n\geq 0} N_n$-mod of categories of finitely generated left N_n-modules categorify the polynomial representation of the Weyl group. (See Section 3.)

Our natural question is how to categorify Virasoro–Magri PVA $\mathcal{V}$, a chiralization of the polynomial algebra. Let Σ_n be the symmetric group on n letters and set $\Sigma = \bigoplus_{n\geq 0} \mathbb{C}\Sigma_n$. We denote by $K_0(\Sigma)$ the Grothendieck group of the category of finitely generated projective modules over Σ. The main result of this paper shows that $K_0(\Sigma)$ is isomorphic to $\mathcal{V}_{\mathbb{Z}}$ as PVA, where $\mathcal{V}_{\mathbb{Z}}$ is the integral form of $\mathcal{V}$ defined in Remark 2.6 (2) (Theorem 4.8).

Recall that the nil-Coxeter algebra is a degenerate homogeneous version of the symmetric group algebra. Another main result of this paper is difference of these two algebras becomes manifest through the Zhu map between Virasoro–Magri PVA and the polynomial algebra $\mathbb{C}[x]$.

Our paper is organized as follows. In Section 1, we review the notion of vertex algebras and the state-field correspondence. In Section 2, we give an explicit description of a