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Stability of depths of powers of edge ideals

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ABSTRACT

Let G be a graph and let $I := I(G)$ be its edge ideal. In this paper, we provide an upper bound of n from which depth $R/I(G)^n$ is stationary, and compute this limit explicitly. This bound is always achieved if G has no cycles of length 4 and every its connected component is either a tree or a unicyclic graph.

Introduction

Let $R = K[x_1, \ldots, x_r]$ be a polynomial ring over a field K and I be a homogeneous ideal in R. Brodmann [2] showed that depth R/I^n is a constant for sufficiently large n. Moreover

$$\lim_{n\to\infty} \operatorname{depth} R/I^n \leqslant \dim R - \ell(I),$$

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where $\ell(I)$ is the analytic spread of I. It was shown in [6, Proposition 3.3] that this is an equality when the associated graded ring of I is Cohen–Macaulay. We call the smallest number n_0 such that $\operatorname{depth} R/I^n = \operatorname{depth} R/I^{n_0}$ for all $n \geqslant n_0$, the *index of depth stability* of I, and denote this number by $\operatorname{dstab}(I)$. It is of natural interest to find a bound for $\operatorname{dstab}(I)$. As until now we only know effective bounds of $\operatorname{dstab}(I)$ for few special classes of ideals I, such as complete intersection ideals (see [5]), square-free Veronese ideals (see [8]), polymatroidal ideals (see [10]). In this paper we will study this problem for *edge ideals*.

From now on, every graph G is assumed to be simple (i.e., a finite, undirected, loopless and without multiple edges) without isolated vertices on the vertex set $V(G) = [r] := \{1, \dots, r\}$ and the edge set $E(G)$ unless otherwise indicated. We associate to G the quadratic squarefree monomial ideal

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \subseteq R = K[x_1, \dots, x_r]$$

which is called the edge ideal of G.

If I is a polymatroidal ideal in R, Herzog and Qureshi proved that $\operatorname{dstab}(I) < \dim R$ and they asked whether $\operatorname{dstab}(I) < \dim R$ for all Stanley–Reisner ideals I in R (see [10]). For a graph G, if every its connected component is nonbipartite, then we can see that $\operatorname{dstab}(I(G)) < \dim R$ from [4]. In general, there is no an absolute bound of $\operatorname{dstab}(I(G))$ even in the case G is a tree (see [20]). In this paper we will establish a bound of $\operatorname{dstab}(I(G))$ for any graph G. In particular, $\operatorname{dstab}(I(G)) < \dim R$.

The first main result of the paper shows that the limit of the sequence $\operatorname{depth} R/I(G)^n$ is the number s of connected bipartite components of G and $\operatorname{depth} R/I(G)^n$ immediately becomes constant once it reaches the value s. Moreover, $\operatorname{dstab}(I(G))$ can be obtained via its connected components.

Theorem 4.4. *Let G be a graph with p connected components $G_1, \dots, G_p$. Let s be the number of connected bipartite components of G. Then*

(1) $\min\{\operatorname{depth} R/I(G)^n \mid n \geqslant 1\} = s$.
(2) $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \operatorname{depth} R/I(G)^n = s\}$.
(3) $\operatorname{dstab}(I(G)) = \sum_{i=1}^{p} \operatorname{dstab}(I(G_i)) - p + 1$.

The second one estimates an upper bound for $\operatorname{dstab}(I(G))$. Before stating our result, we recall some terminologies from graph theory. In a graph G, a *leaf* is a vertex of degree one and a *leaf edge* is an edge incident with a leaf. A connected graph is called a *tree* if it contains no cycles, and it is called a *unicyclic* graph if it contains exactly one cycle. We use the symbols $v(G)$, $\varepsilon(G)$ and $\varepsilon_0(G)$ to denote the number of vertices, edges and leaf edges of G, respectively.

Theorem 4.6. *Let G be a graph. Let $G_1, \dots, G_s$ be all connected bipartite components of G and let $G_{s+1}, \dots, G_{s+t}$ be all connected nonbipartite components of G. Let $2k_i$ be the*