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Profinite groups and the fixed points of coprime automorphisms ☆

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ABSTRACT

The main result of the paper is the following theorem. Let q be a prime and A an elementary abelian group of order q^3. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is locally nilpotent for each $a \in A^{\#}$. Then the group G is locally nilpotent.

1. Introduction

Let A be a finite group acting on a finite group G. Many well-known results show that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of A has influence over the structure of G. The influence is especially strong if $(|A|, |G|) = 1$, that is, the action of A on G is coprime. Let $A^{\#}$ denote the set of non-identity elements of A. The following theorem was proved in [12].

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Theorem 1.1. *Let q be a prime and A an elementary abelian q-group of order at least q^3. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ is nilpotent for each $a \in A^{\#}$. Then G is nilpotent.*

There are well-known examples that show that the above theorem fails if the order of A is q^2. Indeed, let p and r be odd primes and H and K the groups of order p and r respectively. Denote by $A = \langle a_1, a_2 \rangle$ the noncyclic group of order four with generators a_1, a_2 and by Y the semidirect product of K by A such that a_1 acts on K trivially and a_2 takes every element of K to its inverse. Let B be the base group of the wreath product $H \wr Y$ and note that $[B, a_1]$ is normal in $H \wr Y$. Set $G = [B, a_1]K$. The group G is naturally acted on by A and $C_G(A) = 1$. Therefore $C_G(a)$ is abelian for each $a \in A^{\#}$. But, of course, G is not nilpotent.

In [11] the situation of Theorem 1.1 was studied in greater detail and the following result was obtained.

Theorem 1.2. *Let q be a prime and A an elementary abelian q-group of order at least q^3. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ is nilpotent of class at most c for each $a \in A^{\#}$. Then G is nilpotent and the class of G is bounded by a function depending only on q and c.*

Of course, the above results have a bearing on profinite groups. By an automorphism of a profinite group we always mean a continuous automorphism. A group A of automorphisms of a profinite group G is coprime if A has finite order while G is an inverse limit of finite groups whose orders are relatively prime to the order of A. Using the routine inverse limit argument it is easy to deduce from Theorem 1.1 and Theorem 1.2 that if G is a profinite group admitting a coprime group of automorphisms A of order q^3 such that $C_G(a)$ is pronilpotent for all $a \in A^{\#}$, then G is pronilpotent; and if $C_G(a)$ is nilpotent for all $a \in A^{\#}$, then G is nilpotent. Yet, certain results on fixed points in profinite groups cannot be deduced from corresponding results on finite groups. The purpose of the present paper is to establish the following theorem.

Theorem 1.3. *Let q be a prime and A an elementary abelian q-group of order at least q^3. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is locally nilpotent for each $a \in A^{\#}$. Then G is locally nilpotent.*

Recall that a group is locally nilpotent if every finitely generated subgroup is nilpotent. Though Theorem 1.3 looks similar to Theorems 1.1 and 1.2, in fact it cannot be deduced directly from those results. Moreover, the proof of Theorem 1.3 is very much different from those of Theorems 1.1 and 1.2. In particular, unlike the other results, Theorem 1.3 relies heavily on the Lie-theoretical techniques created by Zelmanov in his solution of the restricted Burnside problem [14,15]. The general scheme of the proof of Theorem 1.3 is similar to that of the result in [5].