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Variants of theorems of Baer and Hall on finite-by-hypercentral groups

Carlo Casolo [a], Ulderico Dardano [b,*], Silvana Rinauro [c]

[a] *Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67A, I-50134 Firenze, Italy*
[b] *Dipartimento di Matematica e Applicazioni "R.Caccioppoli", Università di Napoli "Federico II", Via Cintia - Monte S. Angelo, I-80126 Napoli, Italy*
[c] *Dipartimento di Matematica, Informatica ed Economia, Università della Basilicata, Via dell'Ateneo Lucano 10 - Contrada Macchia Romana, I-85100 Potenza, Italy*

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We show that if a group G has a finite normal subgroup L such that G/L is hypercentral, then the index of the (upper) hypercenter of G is at most $|Aut(L)| \cdot |L|$. It follows an explicit bound for $|G/Z_{2m}(G)|$ in terms of $d = |\gamma_{m+1}(G)|$ and independent of $m \in \mathbb{N}$, provided d is finite. This completes recent results generalizing classical theorems by R. Baer and P. Hall. Then we extend other results in the literature by applying our results to groups of automorphisms acting in a restricted way on an ascending normal series of a group G.

* Corresponding author.
E-mail addresses: casolo@math.unifi.it (C. Casolo), dardano@unina.it (U. Dardano), silvana.rinauro@unibas.it (S. Rinauro).

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Introduction and main results

A classical theorem by R. Baer states that, if for some positive integer m the m-th term $Z_m(G)$ of the upper central series of a group G has finite index t in G, then there is a finite normal subgroup L of G such that G/L is nilpotent of class at most m, that is $G/L = Z_m(G/L)$. Moreover, the order of such an L may be bounded by a function $\beta(m,t)$, see the proof of 14.5.1 in [10]. Clearly, one may take L to be the mth-term $\gamma_{m+1}(G)$ of the lower central series of G. Our references for notation and basic facts are [7] and [10].

It is worth noticing that the bound in Baer's Theorem cannot be made dependent on $t = |G/Z_m(G)|$ only. For a prime p and integers $d, m \geq 2$, let F/R be the elementary abelian group of order p^d, as quotient of the free group F on d generators, and $K = [R, {}_mF]$; it is proved in [2] (see also [3]) that if $G = F/K$, then $Z_m(G) = R/K$, hence $|G/Z_m(G)| = p^d$, while $|\gamma_{m+1}(G)| = p^{\chi_m(d)}$ where $\chi_m(d)$ is the rank of the free abelian group $\gamma_m(F)/\gamma_{m+1}(F)$. On the other hand, in Proposition 3 we will observe that for any group G the order of $\gamma_{2m}(G)$ is bounded by a function of $t = |G/Z_m(G)|$ only, provided t is finite.

In the opposite direction, P. Hall showed that, if there is a normal subgroup L of the group G, with finite order d, such that G/L is nilpotent of class at most m, then $G/Z_{2m}(G)$ has finite order bounded by the function $h(m,d) = d^{(\log_2 d)^{2m} + \log_2 d}$ (see [9], page 118). Recently, in [6] it has been proved that $|G/Z_{2m}(G)|$ is finite, and bounded, under the weaker condition that $\bar{G} = G/Z_m(G)$ has a subgroup $\bar{L}$ with finite order d such that G/L is nilpotent of class at most m. In particular, accordingly to Theorem A of [6], there exists a bound for $|G/Z_{2m}(G)|$ in terms of the order of $L = \gamma_{m+1}(G)$ only. However, such bound is not explicitly computed and the proof of its existence is mostly left to the reader.

Considering possibly transfinite central series, recently, in [4] it has been shown that the hypercenter of G has finite index t if and only if there is a finite normal subgroup L with order d such that G/L is *hypercentral*, that is coincides with its hypercenter. Recall that the (upper) *hypercenter* of a group G is the last term of the upper central series of G (see details below). Then in Theorem B of [8] it has been shown that d may be bounded by a function of t, namely $t^{(1+\log_2 t)/2}$. Here we complete the picture by showing that t in turn may be bounded by a function of d.

Theorem A. *If a group G has a finite normal subgroup L such that G/L is hypercentral, then the hypercenter of G has index bounded by* $|\mathrm{Aut}(L)| \cdot |Z(L)|$.

Observe that the bound we have is quite neat, and indeed from this we may rather simply deduce a corollary which shows that the picture in Hall Theorem is not similar to that in Baer Theorem, by giving an explicit bound for $|G/Z_{2m}(G)|$ in terms of $|\gamma_{m+1}(G)|$ only, provided the latter is finite.