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The classification of partition homogeneous groups with applications to semigroup theory

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ABSTRACT

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a *partition* of n, a sequence of positive integers in non-increasing order with sum n. Let $\Omega := \{1, \ldots, n\}$. An ordered partition $P = (A_1, A_2, \ldots)$ of Ω has *type* λ if $|A_i| = \lambda_i$.
Following Martin and Sagan, we say that G is λ-*transitive* if, for any two ordered partitions $P = (A_1, A_2, \ldots)$ and $Q = (B_1, B_2, \ldots)$ of Ω of type λ, there exists $g \in G$ with $A_i g = B_i$ for all i. A group G is said to be λ-*homogeneous* if, given two ordered partitions P and Q as above, inducing the sets $P' = \{A_1, A_2, \ldots\}$ and $Q' = \{B_1, B_2, \ldots\}$, there exists $g \in G$ such that $P'g = Q'$. Clearly a λ-transitive group is λ-homogeneous. The first goal of this paper is to classify the λ-homogeneous groups (Theorems 1.1 and 1.2). The second goal is to apply this classification to a problem in semigroup theory.
Let $\mathcal{T}_n$ and $\mathcal{S}_n$ denote the transformation monoid and the symmetric group on Ω, respectively. Fix a group $H \leqslant \mathcal{S}_n$. Given a non-invertible transformation $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ and a group $G \leqslant \mathcal{S}_n$, we say that (a, G) is an H-*pair* if the semigroups generated by $\{a\} \cup H$ and $\{a\} \cup G$ contain the same non-units, that is, $\langle a, G\rangle \setminus G = \langle a, H\rangle \setminus H$. Using the classification of the λ-homogeneous groups we classify all the $\mathcal{S}_n$-pairs (Theorem 1.8). For a multitude of transformation semigroups this theorem immediately implies a description of

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their automorphisms, congruences, generators and other relevant properties (Theorem 8.5).
This topic involves both group theory and semigroup theory; we have attempted to include enough exposition to make the paper self-contained for researchers in both areas.
The paper finishes with a number of open problems on permutation and linear groups.

1. Introduction

For notation and basic results on group theory we refer the reader to [17,21]; for semigroup theory we refer the reader to [28]. A permutation group G on $\Omega := \{1, \ldots, n\}$ (a subgroup of $\mathcal{S}_n$) is said to be t-transitive if it acts transitively on the set of t-tuples of distinct elements of Ω, and is t-homogeneous if it acts transitively on the set of t-element subsets of $\{1, \ldots, n\}$. Clearly a t-transitive group is t-homogeneous. Livingstone and Wagner [38] showed that the converse is true if $5 \le t \le n/2$; the t-homogeneous, but not t-transitive groups for $t = 2, 3, 4$ were determined by Kantor [29,30].

Martin and Sagan [40] defined the notion of λ-transitivity of a permutation group, as a generalization of t-transitivity, where λ is a partition of n (the degree of the permutation group). In the case where $\lambda = (n - t, 1, 1, \ldots, 1)$, λ-transitivity is just t-transitivity. There is a weakening of this notion, here called λ-homogeneity; this notion is natural in permutation groups and also in transformation semigroups. Our first goal is to investigate the relationship between λ-transitivity and λ-homogeneity.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a *partition* of n, a sequence of positive integers in non-increasing order with sum n. Now let G be a permutation group on Ω, where $|\Omega| = n$. An ordered partition $P = (A_1, A_2, \ldots)$ of Ω has *type* λ if $|A_i| = \lambda_i$.

Following Martin and Sagan, we say that G is λ-*transitive* if, for any two partitions $P = (A_1, A_2, \ldots)$ and $Q = (B_1, B_2, \ldots)$ of Ω of type λ, there exists $g \in G$ with $A_i g = B_i$ for all i. A group G is said to be λ-*homogeneous* if, given two ordered partitions P and Q as above, inducing the sets $P' = \{A_1, A_2, \ldots\}$ and $Q' = \{B_1, B_2, \ldots\}$, there exists $g \in G$ such that $P'g = Q'$. Clearly a λ-transitive group is λ-homogeneous.

Our motivating question is whether the analogue of the Livingstone–Wagner theorem holds for partition homogeneity and transitivity. Indeed, we see that a group of degree n which is t-homogeneous but not t-transitive is λ-homogeneous but not λ-transitive, where λ is the partition $(n - t, 1, 1, \ldots, 1)$ (with t ones).

We now introduce the three main results proved in this paper: Theorems 1.1 and 1.2 on permutation groups, and Theorem 1.8 on transformation semigroups.

Our first main theorem characterizes the λ-homogeneous groups. The partition $\lambda = (1, 1, \ldots, 1)$ is excluded since every permutation group is λ-homogeneous but only the symmetric group is λ-transitive.