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Growth of multiplicities of graded families of ideals

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ABSTRACT

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Let $I_\bullet = \{I_n\}_{n\in\mathbb{N}}$ be a graded family of $\mathfrak{m}$-primary ideals in R. We examine how far off from a polynomial can the length function $\ell_R(R/I_n)$ be asymptotically. More specifically, we show that there exists a constant $\gamma > 0$ such that for all $n \geq 0$,

$$\ell_R(R/I_{n+1}) - \ell_R(R/I_n) < \gamma n^{d-1}.$$

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1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Let $I_\bullet = \{I_n\}_{n \in \mathbb{N}}$ be a graded family of $\mathfrak{m}$-primary ideals in R (that is, $I_0 = R$ and $I_m I_n \subseteq I_{m+n}$ for all $m, n \in \mathbb{N}$). The *volume* of $I_\bullet$ is defined to be

$$\operatorname{vol}(I_\bullet) := \limsup_{n \to \infty} \frac{\ell_R(R/I_n)}{n^d/d!},$$

where $\ell_R(-)$ denotes the length function. Classically, if $I_n = I^n$, for all n, are powers of a fixed $\mathfrak{m}$-primary ideal I then $\operatorname{vol}(I_\bullet) = e(I)$ is the well known Hilbert–Samuel *multiplicity* of I. In recent years, there has been a surge of interest in studying the volume $\operatorname{vol}(I_\bullet)$, and particularly the asymptotic behavior of $\ell_R(R/I_n)$, when $I_\bullet$ is an arbitrary graded family (cf. [1–4,7,9–14,18]). Under mild assumptions on the ring R, the following statements have been established:

(1) $\operatorname{vol}(I_\bullet)$ is an actual limit, i.e., the limit $\lim_{n \to \infty} \dfrac{\ell_R(R/I_n)}{n^d/d!}$ exists; and
(2) $\operatorname{vol}(I_\bullet)$ is the same as the *asymptotic multiplicity* of $I_\bullet$, i.e.,

$$\operatorname{vol}(I_\bullet) = \lim_{s \to \infty} \frac{e(I_s)}{s^d}.$$

This program of research originated from Okounkov's work [15,16], in which the asymptotic multiplicity of graded families of algebraic objects was interpreted in terms of the volume of certain cones (the *Okounkov body*). This method was later developed more systematically by Lazarsfeld and Mustaţă [12] and by Kaveh and Khovanskii [11] for graded linear series on projective schemes. In particular, statements (1) and (2) were proved in [12] when R is essentially of finite type over an algebraically closed field k with $R/\mathfrak{m} = k$. In a series of papers [1–4], Cutkosky used a different approach to extend this result to hold for an arbitrary field k. Specifically, he showed that statement (1) holds for all graded families $I_\bullet$ of $\mathfrak{m}$-primary ideals in R if and only if the nilradical of the $\mathfrak{m}$-adic completion of R has dimension strictly less than d, and statement (2) holds when R is analytically unramifield. As a consequence, it was also deduced (see [3, Corollary 6.3]) that the *epsilon multiplicity* of an ideal I, $\epsilon(I) := \limsup_{n \to \infty} \frac{\ell_R(H^0_{\mathfrak{m}}(R/I^n))}{n^d/d!}$, defined by Ulrich and Validashti [18], existed as an actual limit.

It is known (cf. [3,5]) that the volume $\operatorname{vol}(I_\bullet)$ in general can be an irrational number. Thus, $\ell_R(R/I_n)$ asymptotically does not behave like a polynomial. Our motivation in this paper is the question of how far off from a polynomial can the function $\ell_R(R/I_n)$ be asymptotically. More precisely, we investigate the growth of the difference function $\ell_R(R/I_{n+1}) - \ell_R(R/I_n)$.

It was shown in [1, Theorem 4.5] that when R is a *regular* local ring of dimension $d > 0$ and $I_\bullet = \{I_n\}_{n \in \mathbb{N}}$ is a graded *filtration* (i.e., $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$) of $\mathfrak{m}$-primary