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Exact pairs of homogeneous zero divisors

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ABSTRACT

Let S be a standard graded Artinian algebra over a field k. We identify constraints on the Hilbert function of S which are imposed by the hypothesis that S contains an exact pair of homogeneous zero divisors. As a consequence, we prove that if S is a compressed level algebra, then S does not contain any homogeneous zero divisors.

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In [18], Henriques and Şega defined the pair of elements (a,b) in a commutative ring S to be an *exact pair of zero divisors* if $(0 :_S a) = (b)$ and $(0 :_S b) = (a)$. We take S to be a standard graded Artinian algebra over a field and we identify constraints on the Hilbert function of S which are imposed by the hypothesis that S contains an exact pair (θ_1, θ_2) of homogeneous zero divisors. In Theorem 2.10 we prove that the main numerical constraint depends on the sum $\deg \theta_1 + \deg \theta_2$, but not on the individual numbers $\deg \theta_1$ or $\deg \theta_2$. In other words, the numerical constraint imposed on S by having an exact pair of homogeneous zero divisors of degrees d_1 and d_2 is the same as the constraint imposed by having an exact pair of homogeneous zero divisors of degrees 1 and $d_1 + d_2 - 1$. This result is especially curious because it is possible for S to have an exact pair of homogeneous zero divisors of degrees 2 and 2 without having any homogeneous exact zero divisors of degree 1; see Example 3.1. Our main result is Theorem 2.10.

Theorem 2.10. *Let S be a standard graded Artinian k-algebra. Suppose that (θ_1, θ_2) is an exact pair of homogeneous zero divisors in S. If $D = \deg \theta_1 + \deg \theta_2$, then the Hilbert series of S is divisible by $\frac{t^D-1}{t-1}$.*

In the statement of Theorem 2.10, the algebra S is Artinian, so the Hilbert series, $\mathrm{HS}_S(t)$, of S is a polynomial in $\mathbb{Z}[t]$, the expression $\frac{t^D-1}{t-1}$ is equal to the polynomial $1 + t + t^2 + \cdots + t^{D-1}$ of $\mathbb{Z}[t]$, and

$$\begin{aligned}&\text{“the Hilbert series of } S \text{ is divisible by } \tfrac{t^D-1}{t-1}\text{” means that the polynomial}\\ &\quad 1 + t + t^2 + \cdots + t^{D-1} \ \text{ divides the polynomial } \ \mathrm{HS}_S(t)\\ &\quad \text{in the polynomial ring } \mathbb{Z}[t].\end{aligned} \tag{0.1}$$

We apply Theorem 2.10 in Section 3 to obtain a list of conditions on the standard graded k-algebra S, each of which leads to the conclusion that S does not have an exact pair of homogeneous zero divisors. These results are striking due to the connection between the existence of totally reflexive S-modules and the existence of exact zero divisors in S.

Definition 0.2. Let S be a commutative ring. A finitely generated S-module M is called *totally reflexive* if there exists a doubly infinite sequence of finitely generated free S-modules