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Bar operators for quasiparabolic conjugacy classes in a Coxeter group

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A B S T R A C T

The action of a Coxeter group W on the set of left cosets of a standard parabolic subgroup deforms to define a module $\mathcal{M}^J$ of the group's Iwahori–Hecke algebra $\mathcal{H}$ with a particularly simple form. Rains and Vazirani have introduced the notion of a *quasiparabolic set* to characterize W-sets for which analogous deformations exist; a motivating example is the conjugacy class of fixed-point-free involutions in the symmetric group. Deodhar has shown that the module $\mathcal{M}^J$ possesses a certain antilinear involution, called the bar operator, and a certain basis invariant under this involution, which generalizes the Kazhdan–Lusztig basis of $\mathcal{H}$. The well-known significance of this basis in representation theory makes it natural to seek to extend Deodhar's results to the quasiparabolic setting. In general, the obstruction to finding such an extension is the existence of an appropriate quasiparabolic analogue of the "bar operator." In this paper, we consider the most natural definition of a quasiparabolic bar operator, and develop a theory of "quasiparabolic Kazhdan–Lusztig bases" under the hypothesis that such a bar operator exists. Giving content to this theory, we prove that a bar operator in the desired sense does exist for quasiparabolic W-sets given by twisted conjugacy classes of twisted involutions. Finally, we prove

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several results classifying the quasiparabolic conjugacy classes in a Coxeter group.

Contents

1. Introduction

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$, and let $\mathcal{H} = \mathcal{H}(W, S)$ be its Iwahori–Hecke algebra: this is the $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}$, with a basis given by the symbols H_w for $w \in W$, whose multiplication is uniquely determined by the condition that

$$H_s H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) > \ell(w) \\ H_{sw} + (v - v^{-1}) \cdot H_w & \text{if } \ell(sw) < \ell(w) \end{cases} \qquad \text{for } s \in S \text{ and } w \in W.$$

Observe that H_1 (which we typically write as 1 or omit) is the multiplicative unit of $\mathcal{H}$ and that H_s is invertible for each $s \in S$. There exists a unique ring homomorphism $\mathcal{H} \to \mathcal{H}$ with $v \mapsto v^{-1}$ and $H_s \mapsto H_s^{-1}$; we denote this map by $H \mapsto \overline{H}$, and refer to it as the *bar operator* of $\mathcal{H}$.

Certain representations of W admit natural and interesting deformations to modules of the algebra $\mathcal{H}$. For example, $\mathcal{H}$ viewed as a left module over itself clearly deforms the regular representation of W. For another example, suppose $J \subset S$ is a subset of simple generators and let $X = W/W_J$ be the set of left cosets of the standard parabolic subgroup $W_J = \langle J \rangle$ in W. Define the height of a coset to be the minimal length of any of its elements, i.e., set

$$\mathrm{ht}(\mathcal{C}) = \min_{w \in \mathcal{C}} \ell(w) \qquad \text{for a left coset } \mathcal{C} \in W/W_J.$$