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Symmetry in the core of a zero-dimensional monomial ideal

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A B S T R A C T

The core of an ideal is the intersection of all of its reductions. We have shown that under certain conditions, the exponent set of the core of a zero-dimensional monomial ideal exhibits translational symmetry. In addition, in two dimensions, the core of a monomial ideal is often the core of a reduction number one ideal. We provide an algorithm for obtaining that reduction number one ideal and, subsequently, its core.

1. Introduction

In this paper, we investigate the exponent set of the core of a zero-dimensional monomial ideal. The core of an ideal arises naturally from the study of reductions. Introduced by D.G. Northcott and D. Rees in [10], reductions provide a method for studying the growth of powers of an ideal through simplifications of that ideal. An ideal J contained in I is called a *reduction* of I if $JI^r = I^{r+1}$ for some $r \in \mathbb{N}$. In a Noetherian ring, J is a reduction of I provided I is integral over J.

From the latter characterization, the integral closure, $\overline{I}$, is the unique largest ideal for which I is a reduction. On the other hand, most ideals I have infinitely many minimal reductions [10]. To remedy this lack of uniqueness, we take the intersection of all

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reductions of I. This intersection, first introduced by D. Rees and J. Sally in [13], is called the *core of* I, denoted $\text{core}(I)$. In a polynomial ring, both the integral closure and the core of a monomial ideal are monomial (see for instance [14, 1.4.2] and [3, 5.1]). It is well known that the integral closure of a monomial ideal is determined by the convex hull of its exponent set, the Newton polyhedron $\text{NP}(I)$ (see for instance [14, 1.4]). We seek an analogous description for the core of a monomial ideal.

Previously, Corso, Huneke, Hyry, Polini, Smith, Trung, and Ulrich, and Vitulli [3–8,11,12] have shown that in various settings, including 0-dimensional monomial ideals, $\text{core}(I)$ can be expressed as a colon ideal. Indeed, $\text{core}(I) = J^{t+1} : I^t$, where J is a minimal or locally minimal reduction of I and t is sufficiently large. However, a minimal reduction of a monomial ideal need not be monomial, so $J^{t+1} : I^t$ can be difficult to compute. To address this, Polini, Ulrich, and Vitulli showed $\text{core}(I) = \text{mono}(K)$ for a 0-dimensional monomial ideal I, where K is a general locally minimal reduction of I, and $\text{mono}(K)$ is the largest monomial ideal contained in K [12, 3.6]. Though computationally more effective, this still does not explicitly connect $\text{core}(I)$ to the exponent set of I. In what follows, we give an algorithm for computing $\Gamma(\text{core}(I))$, the exponent set of the core of I, and show that it has translational symmetry with respect to $\text{NP}(I)$.

First, we consider ideals I in d-dimensional polynomial rings that have a d-generated monomial reduction. In this setting, $\text{NP}(I)$ is determined by a single hyperplane. Proposition 2.2 shows that the exponent set of the core, $\Gamma(\text{core}(I))$, is a d-fold translation of $\Gamma(\text{core}(I) : I)$. Monomial almost complete intersection ideals, which may not have a d-generated monomial reduction, have $(d + 1)$-fold symmetry in the exponent sets of their cores, as shown in Proposition 2.4.

We then introduce *local translational symmetry*, or *LTS*, in Definition 3.2 to describe $\Gamma(\text{core}(I))$ when we restrict to $d = 2$ but allow the smallest monomial reduction of a 0-dimensional monomial ideal I to have any number of generators. LTS generalizes the global symmetry described above for ideals whose Newton polyhedron is determined by multiple hyperplanes. This definition proves useful not only for describing the core of I, but also for determining whether or not I has reduction number one (see Theorem 3.7). The latter characterization yields Algorithm 3.9, an efficient method of calculating the smallest reduction number one ideal containing I.

Because not all cores of ideals have the expected symmetry even in dimension 2 (compare Examples 3.12 and 3.14), we need to determine precisely when LTS occurs. To do so, we turn to the coefficient ideal of I, $\text{coef}(I)$. The coefficient ideal, as introduced by Aberbach and Huneke in [2] and described for this setting in [9], is the largest ideal $\mathfrak{a}$ such that $I\mathfrak{a} = J\mathfrak{a}$ for any minimal reduction J of I. Theorem 3.11 shows that a core exhibits LTS if and only if $\text{core}(I) = J\,\text{coef}(I)$, and that this happens only when $\text{core}(I)$ is also the core of a reduction number one ideal. To present Theorem 3.11 in another context, we could say the core exhibits LTS provided it is “expected” in the sense of [15]. Finally, we show that monomial almost complete intersections in $k[x, y]$ are at least one class of ideals which satisfy these equivalent conditions.