Journal of Algebra 446 (2016) 58–76

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Sectional genera of parameter ideals

Shiro Goto [a], Kazuho Ozeki [b,*]

[a] Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
[b] Department of Mathematical Science, Faculty of Science, Yamaguchi University, 1677-1 Yoshida, Yamaguchi 753-8512, Japan

A R T I C L E   I N F O

Article history:
Received 14 April 2014
Available online 30 September 2015
Communicated by Bernd Ulrich

MSC:
13D40
13H15
13H10

Keywords:
Hilbert function
Hilbert coefficient
Homological degrees

A B S T R A C T

Let M be a finitely generated module over a Noetherian local ring. This paper reports, for a given parameter ideal Q for M, a criterion for the equality $\mathrm{g}_s(Q;M) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M) - \mathrm{T}_Q^1(M)$, where $\mathrm{g}_s(Q;M)$, $\mathrm{hdeg}_Q(M)$, $\mathrm{e}_Q^0(M)$, and $\mathrm{T}_Q^1(M)$ respectively denote the sectional genus, the homological degree, the multiplicity, and the homological torsion of M with respect to Q.

Contents

* Corresponding author.
E-mail addresses: goto@math.meiji.ac.jp (S. Goto), ozeki@yamaguchi-u.ac.jp (K. Ozeki).

http://dx.doi.org/10.1016/j.jalgebra.2015.09.005

1. Introduction

The notion of the sectional genera of commutative rings was introduced by A. Ooishi [11], and since then, many authors have been engaged in the development of the theory. The purpose of our paper is to give a criterion for a certain equality of the sectional genera of parameters for modules.

To state the problems and the results of our paper, let us fix some of our terminology. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let M be a finitely generated A-module with $s = \dim_A M$. For simplicity, throughout this paper, we assume that A is $\mathfrak{m}$-adically complete and the residue class field $A/\mathfrak{m}$ of A is infinite. Let I be a fixed $\mathfrak{m}$-primary ideal in A and let $\ell_A(N)$ denote, for an A-module N, the length of N. Then there exist integers $\{\mathrm{e}_I^i(M)\}_{0 \le i \le s}$ such that

$$\ell_A(M/I^{n+1}M) = \mathrm{e}_I^0(M)\binom{n+s}{s} - \mathrm{e}_I^1(M)\binom{n+s-1}{s-1} + \cdots + (-1)^s \mathrm{e}_I^s(M)$$

for all $n \gg 0$. We call $\mathrm{e}_I^i(M)$ the i-th Hilbert coefficient of M with respect to I and especially call the leading coefficient $\mathrm{e}_I^0(M)$ (> 0) the multiplicity of M with respect to I. We set

$$\mathrm{g}_s(I; M) = \ell_A(M/IM) - \mathrm{e}_I^0(M) + \mathrm{e}_I^1(M)$$

and call it the sectional genus of M with respect to I.

In this paper we need the notions of homological degrees and torsions of modules. For each $j \in \mathbb{Z}$ we set

$$M_j = \mathrm{Hom}_A(\mathrm{H}_\mathfrak{m}^j(M), E),$$

where $E = \mathrm{E}_A(A/\mathfrak{m})$ denotes the injective envelope of $A/\mathfrak{m}$ and $\mathrm{H}_\mathfrak{m}^j(M)$ the jth local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$. Then M_j is a finitely generated A-module with $\dim_A M_j \le j$ for all $j \in \mathbb{Z}$ (Fact 2.1).

The homological degree $\mathrm{hdeg}_I(M)$ of M with respect to I is inductively defined in the following way, according to the dimension $s = \dim_A M$ of M.

Definition 1.1. (See [16].) For each finitely generated A-module M with $s = \dim_A M$, we set

$$\mathrm{hdeg}_I(M) = \begin{cases} \ell_A(M) & \text{if } s \le 0, \\ \mathrm{e}_I^0(M) + \sum_{j=0}^{s-1} \binom{s-1}{j} \mathrm{hdeg}_I(M_j) & \text{if } s > 0 \end{cases}$$

and call it the homological degree of M with respect to I.

The homological torsion of M with respect to I is defined as follows.