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A note on extending actions of infinitesimal group schemes

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ABSTRACT

We prove that iterative derivations on projective line cannot be expanded to iterative Hasse–Schmidt derivations, in the case when the iterativity rule is given by a non-algebraic formal group.

1. Introduction

This paper deals with the problem whether for a derivation ∂, there is a *Hasse–Schmidt derivation* $(\partial_n)_{n\in\mathbb{N}}$ such that $\partial = \partial_1$. We recall that a Hasse–Schmidt derivation on a ring R (see [19]) is a sequence

$$\mathbb{D} = (D_i : R \to R)_{i\in\mathbb{N}}$$

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satisfying the following properties:

- $D_0 = \mathrm{id}_R$,
- each D_i is additive,
- for any $x, y \in R$ we have

$$D_i(xy) = \sum_{j+k=i} D_j(x)D_k(y).$$

A Hasse–Schmidt derivation $\mathbb{D}$ is *iterative* if for all $i, j \in \mathbb{N}$ we have

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j}.$$

For a function $f : R \to R$ and a natural number n, we denote by $f^{(n)}$ the n-th compositional power of f. It is well-known that any derivation ∂ on a $\mathbb{Q}$-algebra expands (uniquely) to an iterative Hasse–Schmidt derivation by the formula $\left(\frac{\partial^{(i)}}{i!}\right)_{i\in\mathbb{N}}$. Matsumura proved that the same (without uniqueness) is still true in the case of fields of positive characteristic p for derivations satisfying the (necessary) condition $\partial^{(p)} = 0$ [16]. In Matsumura's terminology, such a derivation ∂ is (*strongly*) *integrable*.

One may wonder why to consider the iterativity condition of such a specific form (although the characteristic 0 example gives a rather strong motivation). It was noticed by Matsumura that actually the iterativity condition as above is governed by the additive group law $X + Y$. Since in characteristic 0, any (one-dimensional) formal group is isomorphic to the additive one, it is a good choice indeed. However, in the case of positive characteristic there are many more formal group laws and it is an interesting question whether the corresponding derivations are integrable. A multiplicative version of Matsumura's theorem was proved by Tyc [20], where the condition $\partial^{(p)} = 0$ is replaced with the (necessary again) condition $\partial^{(p)} = \partial$.

In [10], the authors deal with a more general problem whether iterative m-*truncated* (m is a positive integer) Hasse–Schmidt derivations are integrable. An iterative m-truncated Hasse–Schmidt derivation is a sequence $(\partial_i)_{i<p^m}$ satisfying the higher Leibnitz rules and the appropriate iterativity conditions, see [10, Def. 2.11]. Since a one-truncated additively iterative Hasse–Schmidt derivation is equivalent to a standard derivation ∂ satisfying the condition $\partial^{(p)} = 0$ (similarly in the multiplicative case, where the necessary condition is $\partial^{(p)} = \partial$), this is a natural generalization. In [10], we extend the results of Matsumura and Tyc to the case of an arbitrary truncation (in the additive case, such a generalization is implicit in the work of Ziegler [22]). A certain class of higher-dimensional commutative affine algebraic groups is treated by the first author in [8]. In this paper, we focus on the one-dimensional case and we comment briefly on the higher-dimensional cases in Section 3.1.

We abbreviate the term "additively (resp. multiplicatively) iterative Hasse–Schmidt derivation" by "$\mathbb{G}_a$-derivation" (resp. "$\mathbb{G}_m$-derivation"). Similarly, for any formal group