Journal of Algebra 446 (2016) 346–372

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Tchebotarev theorems for function fields

Sara Checcoli [a,*], Pierre Dèbes [b]

[a] *100, rue des Maths, BP74, 38402 Saint-Martin-d'Hères Cedex, France*
[b] *Laboratoire Paul Painlevé, U.F.R. de Mathématiques, Université Lille 1, 59655 Villeneuve d'Ascq Cedex, France*

A R T I C L E   I N F O

Article history:
Received 16 June 2014
Communicated by Eva Bayer-Fluckiger

MSC:
primary 11R58, 12F10, 12E25, 12E30
secondary 14Gxx, 14E20

Keywords:
Specialization of Galois extensions
Function fields
Tchebotarev property
Hilbert's irreducibility theorem
Local and global fields

A B S T R A C T

The central theme of the paper is the specialization of algebraic function field extensions. Our main results are Tchebotarev type theorems for Galois function field extensions, finite or infinite, over various base fields: under some conditions, we extend the classical finite field case to number fields, p-adic fields, PAC fields, function fields $\kappa(x)$, etc. We also compare the Tchebotarev conclusion – existence of unramified local specializations with Galois group any cyclic subgroup of the generic Galois group (up to conjugation) – to the Hilbert specialization property. For a function field extension with the Tchebotarev property, the exponent of the Galois group is bounded by the l.c.m. of the local specialization degrees. Local–global questions arise for which we provide answers, examples and counter-examples.

1. Introduction

Let K be a field, B a smooth projective and geometrically integral K-variety and $F/K(B)$ a Galois extension of group G, finite or infinite. For every overfield $k \supset K$ and each point $t_0 \in B(k)$, there is a notion of k-specialization of $F/K(B)$ at t_0. For t_0

* Corresponding author.
E-mail addresses: sara.checcoli@gmail.com (S. Checcoli), Pierre.Debes@univ-lille1.fr (P. Dèbes).

http://dx.doi.org/10.1016/j.jalgebra.2015.08.020

not in the branch locus of the extension $F/K(B)$, it is a Galois extension $(Fk)_{t_0}/k$ and the Galois group $\mathrm{Gal}((Fk)_{t_0}/k)$ identifies to some subgroup of G (well-defined up to conjugation by elements of G).

The leading question in this context consists in comparing the Galois groups of the specializations with the "generic" Galois group G. A classical tool is the *Hilbert specialization property*: essentially, a finite extension $F/K(B)$ has the Hilbert specialization property if the "special" groups equal G for "many" specializations over $k = K$.

In this paper, we introduce another specialization property, the *Tchebotarev existence property*, which is, for finite extensions, a function field analog of the existence part in the Tchebotarev density theorem for number fields. We say that $F/K(B)$ has the *Tchebotarev existence property* if for every cyclic subgroup H of G, there exists a local field k over K and a point $t_0 \in B(k)$ such that the specialization $(Fk)_{t_0}/k$ is cyclic, unramified and its Galois group is conjugate to H in G (see Definition 2.5 for more details).

A first motivation to consider this property is the following: for an extension with the Tchebotarev existence property certain local behaviors encode informations on the structure of the Galois group of the extension and vice-versa, as explained later. This provides a function field analog to some results originally obtained over number fields by S. Checcoli and U. Zannier [6].

Compared to the Hilbert property, our property allows more general base fields and base varieties and it is also defined for infinite extensions. Moreover, even if it only preserves the "local" structures, it still encapsulates a good part of the Hilbert property: for example, over $\mathbb{Q}$, Hilbert essentially follows from the Tchebotarev property merely conjoined with the Artin–Whaples theorem; actually, the Hilbert property is somehow squeezed between two variants of the Tchebotarev property (see Proposition 4.6).

The main results of this paper are some "Tchebotarev theorems for function fields" (Theorem 3.2 and Corollary 3.7), which provide concrete situations where the property holds. As a special case, we have the following:

Theorem 1.1. *A Galois extension $F/K(T)$ with F/K regular*[1] *has the Tchebotarev existence property if K is a number field, or a finite field, or a PAC field*[2] *with cyclic extensions of any degree, or a rational function field $\kappa(x)$ with κ a finite field of prime-to-$|G|$ order.*

With some extra good reduction condition on $F/K(T)$, the property is also shown to hold if K is a p-adic field or a formal Laurent series field with coefficients in a finite field, etc. Fields of the form $K = k((\theta))(x)$ are also considered in Theorem 3.10. To our knowledge only the finite field case was covered in the literature. The main ingredients in our proofs are the *twisting lemma* and the *local specialization result* of P. Dèbes

[1] *I.e.* $F \cap \overline{K} = K$ as recalled in Definition 3.1.
[2] Pseudo Algebraically Closed; the definition is recalled in Section 3.1.2 (a).