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Duality of multiple root loci

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The multiple root loci among univariate polynomials of degree n are indexed by partitions of n. We study these loci and their conormal varieties. The projectively dual varieties are joins of such loci where the partitions are hooks. Our emphasis lies on equations and parametrizations that are useful for Euclidean distance optimization. We compute the ED degrees for hooks. Among the dual hypersurfaces are those that demarcate the set of binary forms whose real rank equals the generic complex rank.

1. Introduction

Univariate polynomials of degree n correspond to points in a projective space $\mathbb{P}^n$. The *multiple root locus* Δ_λ associated with a partition $\lambda = (\lambda_1, \dots, \lambda_d)$ is the subvariety of $\mathbb{P}^n$ given by polynomials that have d distinct roots with multiplicities $\lambda_1, \dots, \lambda_d$. The dimension of Δ_λ is d. The singular locus of Δ_λ is the union of certain codimension one subloci Δ_μ, as described in [15, §3]. The degree of Δ_λ was determined by Hilbert in [11].

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He showed that

$$\deg(\Delta_\lambda) = \frac{d!}{m_1! m_2! \cdots m_p!} \cdot \lambda_1 \lambda_2 \cdots \lambda_d, \tag{1}$$

where m_j denotes the number of parts λ_i in the partition λ that are equal to the integer j.

The multiple root loci Δ_λ have been studied in a wide range of contexts and under various different names: coincident root loci [5,9,15], pejorative manifolds [16,23], strata of the discriminant [11,14,18], λ-Chow varieties [19], factorization manifolds [24, Definition 5.2.4], etc. Our motivation arose from the desire to understand the geometry of a model selection problem considered at the interface of symbolic computation [13] and numerical analysis [23]: given a univariate polynomial h, identify a low-dimensional Δ_λ such that h is close to Δ_λ.

Finding a point in Δ_λ that is closest to a given h is a problem of polynomial optimization [4]. We here characterize the geometric duality that underlies this optimization problem, in the sense of [4, Chapter 5]. The key player is the dual variety $(\Delta_\lambda)^\vee$. This variety lives in the dual projective space $\mathbb{P}^n$ and it parametrizes all hyperplanes that are tangent to Δ_λ.

The duals to multiple root loci were studied by Oeding in [19]. He shows that $(\Delta_\lambda)^\vee$ is a hypersurface if and only if $m_1 = 0$, i.e. all parts of λ satisfy $\lambda_i \geq 2$. In [19, Theorem 5.3] an explicit formula is given for the degree of the polynomial that cuts out this hypersurface:

$$\deg\big((\Delta_\lambda)^\vee\big) = \frac{(d+1)!}{m_2! \cdots m_p!} \cdot (\lambda_1 - 1)(\lambda_2 - 1) \cdots (\lambda_d - 1). \tag{2}$$

For the application to optimization in [4, Theorem 5.23], this is the number of complex critical points one encounters when minimizing a general linear function over an affine chart of Δ_λ. For instance, consider $n = 5$ and $\lambda = (3, 2)$. Following Example 4.2 and [6, §3], the polynomial for $(\Delta_\lambda)^\vee$ is the *apple invariant* of degree 12. So, optimizing a linear function over quintics with a triple root and a double root leads to solving an equation of degree 12.

The present paper is a continuation of the studies by Hilbert [11] and Oeding [19]. It is organized as follows. In Section 2 we set up notation and basics. In Theorem 2.5 we parametrize the conormal variety Con_λ that links Δ_λ and $(\Delta_\lambda)^\vee$. Theorem 2.9 offers a parametrization for the projective duals of multiple root loci. These results are derived from the apolarity theory for binary forms, as described in the book by Iarrobino and Kanev [12].

In Section 3 we study the multidegree of the conormal variety Con_λ, and we summarize what is known about the ideals of Δ_λ and $(\Delta_\lambda)^\vee$. Table 1 offers a census of small