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A remark on the Schur multiplier of nilpotent Lie algebras

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In this article, we indicate that the Schur multiplier of every nilpotent Lie algebra of finite dimension at least 2 is non-zero. Also, we present a criterion for nilpotent Lie algebras lacking any covers with respect to the variety of nilpotent Lie algebras of class at most c.

1. Introduction

All Lie algebras are considered over a fixed field Λ *and the square bracket* $[\ ,\ ]$ *denotes the Lie bracket.* The theory of the Schur multiplier and covers for (finite dimensional) Lie algebras has been developed analogously to that for (finite) groups, and proved to

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be a powerful tool in the study of certain types of problems in Lie algebra theory, such as the classification of finite nilpotent Lie algebras; see [3,4,8,9,16]. In broad outline, the above theories run parallel, however some results already obtained are very different. For instance, it is well known that a finite group can admit more than one non-isomorphic cover [6]. But it has been established in [2] that every finite dimensional Lie algebra, up to isomorphism, has a unique cover. In this article, we present another marked difference between these theories. In fact, we show that the Schur multiplier of each nilpotent Lie algebra of finite dimension at least 2, is non-zero. This result does not hold, in general, for nilpotent groups. For example, the Schur multiplier of the quaternion group $Q_n = \langle a, b \mid a^2 = b^{2^{n-2}} = (ab)^2 \rangle$ of order 2^n is trivial [6, Example 2.4.8]. However, note that p-groups in which all non-identity elements have order p always have non-trivial Schur multiplier which forecasts the present result (see [5,6,14,15] for more information on the Schur multipliers of p-groups).

2. Main result

Let L be a Lie algebra presented as the quotient of a free Lie algebra F by an ideal R. Then the *c-nilpotent multiplier* of L, $c \geq 1$, is defined to be the abelian Lie algebra

$$\mathcal{M}^{(c)}(L) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F),$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-th term of the lower central series of F and $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ (see [10,11]). The Lie algebra $\mathcal{M}^{(1)}(L) = \mathcal{M}(L)$ is the most studied *Schur multiplier* of L. One may check that $\mathcal{M}^{(c)}(L)$ is independent of the choice of the free presentation of L. Furthermore, if we set $\gamma^*_{c+1}(L) = \gamma_{c+1}(F)/\gamma_{c+1}(R, F)$, then it is deduced, from the short exact sequence $0 \longrightarrow \mathcal{M}^{(c)}(L) \longrightarrow \gamma^*_{c+1}(L) \longrightarrow \gamma_{c+1}(L) \longrightarrow 0$ and the invariance of $\mathcal{M}^{(c)}(L)$, that $\gamma^*_{c+1}(L)$ is an invariant of L.

The following results shorten the proof of our main theorem.

Lemma 2.1. *Let L be a Lie algebra and $c \geq 1$. Then*

(i) $\gamma^*_{c+1}(L) = 0$ *if and only if L is nilpotent and* $\mathcal{M}^{(c)}(L) = 0$.
(ii) *If $\gamma^*_{c+1}(L) = 0$, then for any ideal N of L, $\gamma^*_{c+1}(L/N) = 0$.*

Proof. (*i*) Straightforward.

(*ii*) Let $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ be a free presentation of L. It is obvious that $L = \overline{\pi}(F/\gamma_{c+1}(R, F))$, in which $\overline{\pi}$ is the natural epimorphism induced by π. Using the assumption, we have $\gamma_{c+1}(R, F) = \gamma_{c+1}(F)$, implying that the factor Lie algebra $F/\gamma_{c+1}(R, F)$ is nilpotent of class c. Therefore, $L = \overline{\pi}(Z_c(F/\gamma_{c+1}(R, F)))$. The result now follows from [11, Corollary 2.2 and Proposition 2.3]. □