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Composition factors of tensor products of symmetric powers

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A R T I C L E I N F O

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A B S T R A C T

We determine the composition factors of the tensor product $S(E) \otimes S(E)$ of two copies of the symmetric algebra of the natural module E of a general linear group over an algebraically closed field of positive characteristic. Our main result may be regarded as a substantial generalisation of the tensor product theorem of Krop, [6], and Sullivan, [8] on composition factors of $S(E)$. We earlier answered the question of which polynomially injective modules are infinitesimally injective in terms of the "divisibility index". We are now able to give an explicit description of the divisibility index for polynomial modules for general linear groups of degree at most 3.

Introduction

This paper is a continuation of [4]. We are interested in the set of composition factors of the m-fold tensor product $S(E)^{\otimes m}$, of the symmetric algebra $S(E)$ of the natural module E for the general linear group $\mathrm{GL}_n(K)$ of degree n, over an algebraically closed field K of characteristic $p > 0$. In [4] we related this to the set of composition factors of $\bar{S}(E)^{\otimes m}$,

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where $\bar{S}(E)$ is the truncated symmetric algebra on E. The main result, Theorem 6.5, of [4], is an explicit description of the set of composition factors of $\bar{S}(E)^{\otimes m}$. Here we use this, in the case $m = 2$, to give an explicit description of the composition factors of $S(E) \otimes S(E)$. The description of a composition factor is as a twisted tensor product of "primitive" modules and may be regarded as a generalisation of the tensor product theorem of Krop, [6], and Sullivan, [8], on the composition factors of $S(E)$.

The layout of the paper is the following. In Section 1 we record, in the general context, some properties of bounded, special and good partitions. Section 2 is the technical heart of the paper. In this we determine the $(2,1)$-special partitions. The importance for us is to have control over the composition factors of $L \otimes \bigwedge^j E$, $j \geq 0$, where L is a composition factor of $\bar{S}(E) \otimes \bar{S}(E)$. The result, Theorem 2.10, is that a partition is $(2,1)$-special if and only if it has the form $\mu + \omega_s$, for some $s \geq 0$, where μ is 2-special and $\omega_s = 1^s$ for $s \geq 1$ and $\omega_0 = 0$. This is key to the tensor product description of a composition factor of $S(E) \otimes S(E)$ that we obtain in Section 3. The arguments of Section 2 are highly inductive and somewhat lengthy, involving repeated application of node removal from Young diagrams. In Section 3 we also give an explicit description, by highest weight, of the polynomial injective modules for $\mathrm{GL}_3(K)$ which are injective on restriction to the first infinitesimal subgroup. This is obtained by combining a criterion from [3] with our description of the 2-good partitions, in the special case $n = 3$.

1. Generalities on bounded, special and good partitions

We use the notation and terminology of [4]. Thus K denotes an algebraically closed field of characteristic $p > 0$. We write E for the natural module for the general linear group $\mathrm{GL}_n(K)$. We write $S(E)$ for the symmetric algebra on E and $\bar{S}(E)$ for $S(E)/I$, where I is the ideal generated by x^p, $x \in E$. Then $\mathrm{GL}_n(K)$ acts naturally on $S(E)$ and $\bar{S}(E)$ as algebra automorphisms. We write $\Lambda^+(n)$ for the set of all partitions of length at most n and for $\lambda \in \Lambda^+(n)$ write $L(\lambda)$ for the irreducible polynomial $\mathrm{GL}_n(K)$-module with highest weight λ. For $1 \leq m \leq n$ we say that $\lambda \in \Lambda^+(n)$ is m-good (resp. m-special) if $L(\lambda)$ is a composition factor of $S(E)^{\otimes m}$ (resp. $\bar{S}(E)^{\otimes m}$).

We use the notation E_n and $L_n(\lambda)$ for E and $L(\lambda)$ when we wish to emphasise the role of n.

We shall need some additional terminology.

Definition 1.1. Let $a, b \geq 0$. We shall say that a partition λ is (a,b)-bounded if $\lambda_{a+1} \leq b$.

Thus a partition λ is (a,b)-bounded if its diagram fits inside the diagram of a partition of the form $r^a b^s$, for some $r, s \geq 1$. For example, $(1,1)$-bounded partitions are hook partitions and $(2,0)$-bounded partitions are those with at most two rows. Note that in general a partition λ is (a,b)-bounded if and only if the transpose λ' is (b,a)-bounded.