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Finite Gröbner–Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids

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A B S T R A C T

This paper shows that every Plactic algebra of finite rank admits a finite Gröbner–Shirshov basis. The result is proved by using the combinatorial properties of Young tableaux to construct a finite complete rewriting system for the corresponding Plactic monoid, which also yields the corollaries that Plactic monoids of finite rank have finite derivation type and satisfy the homological finiteness properties left and right FP_∞. Also, answering a question of Zelmanov, we apply this rewriting system and other techniques to show that Plactic monoids of finite rank are biautomatic.

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Young tableau
Automatic monoids

1. Introduction

The Plactic monoid has its origins in work of Schensted [1] and Knuth [2] concerning certain combinatorial problems and operations on Young tableaux. It was later studied in depth by Lascoux and Schützenberger [3] and has since become an important tool in several aspects of representation theory and algebraic combinatorics; see [4,5]. The first significant application of the Plactic monoid was to the Littlewood–Richardson rule for Schur functions. This is explained in detail in the appendix to the second edition of J.A. Green's influential monograph on the representation theory of the general linear group [6]. The Littlewood–Richardson rule [7] is one of the most important results in the theory of symmetric functions. It provides a combinatorial rule for expressing a product of two Schur functions as a linear combination of Schur functions. Since Schur functions in n variables are the irreducible polynomial characters of $GL_n(\mathbb{C})$, the Littlewood–Richardson rule gives a tensor product rule for $GL_n(\mathbb{C})$. One of the most enlightening proofs of the Littlewood–Richardson rule (see [5, Section 5.4]) is given by lifting the calculus of the Schur function to the integral monoid ring of the Plactic monoid (called the tableau ring; see [4, Chapter 2]).

Subsequently the Plactic monoid has been found to have applications in a range of areas including a combinatorial description of Kostka–Foulkes polynomials [3,8], and to Kashiwara's theory of crystal bases [9,10] leading to the definition of Plactic algebras associated to all classical simple Lie algebras [11–13]. Further results on Robinson–Schensted correspondence and the Plactic relations may be found in [9,14]. Several variations and generalizations of the Plactic monoid have been proposed and investigated including hypoplactic monoids [13], and shifted Plactic monoids [15]. In [16] it is shown that the Hilbert series of the Plactic monoid is given by the Schur–Littlewood formula, and that there are exactly three families of ternary monoids with this Hilbert series. Schützenberger [17] argues that the Plactic monoid ought to be considered as "one of the most fundamental monoids in algebra". He cites three reasons for his own personal "weakness" for the Plactic monoid, the first of them being the application to symmetric functions mentioned above.

Various aspects of the corresponding semigroup algebras, the Plactic algebras, have been investigated; see, for example, [18,19]. These algebras are important special cases in the more general study of algebras defined by homogeneous semigroup presentations [20]. Frequently, fundamental problems about such semigroup algebras require detailed analysis of the corresponding semigroups. An important example of this is given by the theory of Gröbner–Shirshov bases. Kubat & Okniński showed that the Plactic algebra of rank 3 has a finite Gröbner–Shirshov basis [21, Theorem 1] and that Plactic algebras of rank 4 or more do not admit a finite Gröbner–Shirshov basis with respect to the degree-lexicographic ordering over the usual generating set for the Plactic monoid