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Decomposing modular coinvariants

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ABSTRACT

We consider the ring of coinvariants for a modular representation of a cyclic group of prime order p. We show that the classes of the terminal variables in the coinvariants have nilpotency degree p and that the coinvariants are a free module over the subalgebra generated by these classes. An incidental result we have is a description of a Gröbner basis for the Hilbert ideal and a decomposition of the corresponding monomial basis for the coinvariants with respect to the monomials in the terminal variables.

Introduction

Let V denote a finite dimensional representation of a finite group G over a field $\mathbf{F}$. The induced action on the dual V^* extends as degree preserving algebra automorphisms to the symmetric algebra $S(V^*)$ which we denote by $\mathbf{F}[V]$. The ring of invariant polynomials $\mathbf{F}[V]^G := \{f \in \mathbf{F}[V] \mid g(f) = f \ \forall g \in G\}$ is a graded finitely generated subalgebra. The Hilbert ideal I is the ideal $\mathbf{F}[V]^G_+ \cdot \mathbf{F}[V]$ in $\mathbf{F}[V]$ generated by invariants of positive degree. In this paper we study the ring of coinvariants which is the quotient ring

$$\mathbf{F}[V]_G := \mathbf{F}[V]/I.$$

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Since G is finite, $\mathbf{F}[V]_G$ is a finite dimensional vector space. Note also that coinvariants are naturally a G-module as I is closed under the action of G. Coinvariants often contain information about the invariants. For instance, if $|G|$ is a unit in $\mathbf{F}$, then $\mathbf{F}[V]^G$ is polynomial if and only if $\mathbf{F}[V]_G$ satisfies Poincaré duality, see [4,6,13]. Even in the modular case, that is when $|G|$ is divisible by the characteristic of $\mathbf{F}$, some weaker versions of this equivalence still hold in small dimensions see [8,12]. Since $\mathbf{F}[V]_G$ is a finite dimensional vector space, the largest degree of a non-zero component is also finite. It is well known that in the non-modular case the top degree is bounded by the group order but it can be arbitrarily large otherwise [7]. In many modular cases the top degree of the coinvariants also coincides with the maximum degree of an indecomposable invariant and getting efficient bounds for the top degree has been critical when studying effective generation of invariant rings, see for example [5,10].

We now fix our setup. Let p be a prime integer and $\mathbf{F}$ a field of characteristic p. For the rest of the paper, G denotes the cyclic group of prime order p. Fix a generator σ of G. There are exactly p indecomposable G-modules $V_1, \dots, V_p$ over $\mathbf{F}$ and each indecomposable module V_i is afforded by a Jordan block of dimension i with 1's on the diagonal. Let V be an arbitrary G-module over $\mathbf{F}$. Assume that V has l summands so we can write $V = \sum_{1 \le j \le l} V_{n_j}$. Since adding a trivial summand to a module does not affect the coinvariants we assume as well that $n_j > 1$ for $1 \le j \le l$. We set $\mathbf{F}[V] = \mathbf{F}[X_{i,j} \mid 1 \le i \le n_j,\ 1 \le j \le l]$ and the action of σ is given by $\sigma(X_{i,j}) = X_{i,j} + X_{i-1,j}$ for $1 < i \le n_j$ and $\sigma(X_{1,j}) = X_{1,j}$. Following the established convention, we call the variables $X_{n_j,j}$ for $1 \le j \le l$ terminal variables. We use graded reverse lexicographic order on the monomials in $\mathbf{F}[V]$ with $X_{1,j} < \cdots < X_{n_j,j}$. We denote the leading monomial of a polynomial f by $\mathrm{LM}(f)$. We also use lower case letters to denote the images of the variables in the coinvariants.

Certain examples of coinvariants for modular representations of G are studied in [9] and for each example, among other things, a reduced Gröbner basis for the Hilbert ideal and the corresponding monomial basis for the coinvariants are given. The goal of this paper is to demonstrate that some of the properties of the coinvariants and the Hilbert ideal that are identified in the cases studied in that source hold in general for an arbitrary module V. We show that the Hilbert ideal I is generated by the orbit products of $X_{n_j,j}$ for $1 \le j \le l$ (which we denote by $N(X_{n_j,j})$) and polynomials in $A := \mathbf{F}[X_{i,j} \mid 1 \le i \le n_j - 1,\ 1 \le j \le l]$. Notice that $\mathrm{LM}(N(X_{n_j,j})) = X_{n_j,j}^p$. Therefore we get that, apart from the monomials $X_{n_j,j}^p$ for $1 \le j \le l$, no monomial in the unique minimal generating set for the lead term ideal of I is divisible by any $X_{n_j,j}$ for $1 \le j \le l$. We go on to prove that there is an isomorphism of graded $\mathbf{F}$-algebras

$$\mathbf{F}[x_{n_1,1}, \dots, x_{n_l,l}] \cong \mathbf{F}[t_1, \dots, t_l]/\big(t_1^p, \dots, t_l^p\big),$$

where $t_1, \dots, t_l$ are independent variables. Moreover, we show that $\mathbf{F}[V]_G$ is a free module over $\mathbf{F}[x_{n_1,1}, \dots, x_{n_l,l}]$.

For more background on modular invariant theory we refer the reader to [2] and [3].