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Simple irreducible subgroups of exceptional algebraic groups

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ABSTRACT

A closed subgroup of a semisimple algebraic group is called irreducible if it lies in no proper parabolic subgroup. In this paper we classify all irreducible subgroups of exceptional algebraic groups G which are connected, closed and simple of rank at least 2. Consequences are given concerning the representations of such subgroups on various G-modules: for example, with one exception, the conjugacy classes of irreducible simple connected subgroups of rank at least 2 are determined by their composition factors on the adjoint module of G.

1. Introduction

Let G be a reductive connected algebraic group. A subgroup X of G is called *G-irreducible* (or just *irreducible* if G is clear from the context) if it is closed and not contained in any proper parabolic subgroup of G. This definition, due to Serre in [27], generalises the standard notion of an irreducible subgroup of $GL(V)$. Indeed, if $G = GL(V)$, a subgroup X is G-irreducible if and only if X acts irreducibly on V. Similarly, the notion of complete reducibility can be generalised (see [27]): a subgroup X of G is said to be *G-completely reducible* (or *G-cr* for short) if, whenever it is contained in a parabolic subgroup of G, it is contained in a Levi subgroup of that parabolic.

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Now let G be a connected semisimple group. In [20], Liebeck and Testerman studied connected G-irreducible subgroups for the first time, showing amongst other things that they are semisimple and only have a finite number of overgroups in G. Connected G-irreducible subgroups play an important role in determining both the G-cr and non-G-cr connected subgroups of G. The G-cr subgroups of G are simply the L'-irreducible subgroups of L' for each Levi subgroup L of G (noting that G is a Levi subgroup of itself). To determine the non-G-cr subgroups of G one strategy is as follows. Let P be a proper parabolic subgroup with unipotent radical Q and Levi complement L. Then for each L'-irreducible subgroup X, determine the complements to Q in XQ that are not Q-conjugate to X (if any exist). Any non-G-cr connected subgroup will be of this form for some L'-irreducible connected subgroup X.

We now restrict our attention further by letting G be a simple algebraic group of exceptional type over an algebraically closed field K of characteristic p (setting $p = \infty$ for characteristic 0). In this paper we classify the simple, connected G-irreducible subgroups of rank at least 2. For $G = G_2$ this is a trivial consequence of [12, Theorem 1] and for $G = F_4$ this has already been done [32, Theorem 4]. We therefore only have to deal with $G = E_6, E_7$ and E_8, for which we prove the following three theorems. The tables referred to in the statements can be found in Section 10 of the paper.

Theorem 1. *Suppose X is a simple, connected, irreducible subgroup of E_6 of rank at least 2. Then X is* $\mathrm{Aut}(E_6)$*-conjugate to exactly one subgroup of Table 5.*

Theorem 2. *Suppose X is a simple, connected, irreducible subgroup of E_7 of rank at least 2. Then X is E_7-conjugate to exactly one subgroup of Table 6.*

Theorem 3. *Suppose X is a simple, connected, irreducible subgroup of E_8 of rank at least 2. Then X is E_8-conjugate to exactly one subgroup of Table 7.*

We note that Amende [1] covers the G-irreducible subgroups of rank 1 in $G = G_2$, F_4, E_6 and E_7. The semisimple (non-simple) G-irreducible subgroups and the irreducible subgroups of E_8 of rank 1 will be covered in forthcoming work of the author. Also, under various assumptions on the characteristic ($p > 7$ covers all of them), Theorems 1–3 can be deduced from the results in [14]. Our contribution is to remove these characteristic restrictions.

Each subgroup in Tables 5–7 is described by its embedding in some maximal connected subgroup, given in Theorem 3.1. Notation for the embeddings is given in Section 2.

From these results we can prove a number of representation-theoretic corollaries. For the first of these, we need the following definition. Let G be a simple algebraic group (of arbitrary type), V be a module for G and X and Y be subgroups of G. Then we say X and Y share the same composition factors on V if there exists a morphism from X to Y, which is an isomorphism of abstract groups sending the composition factors of X to composition factors of Y.