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New approaches to plactic monoid via Gröbner–Shirshov bases [☆]

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We give two explicit (quadratic) presentations of the plactic monoid in row and column generators correspondingly. Then we give direct independent proofs that these presentations are Gröbner–Shirshov bases of the plactic algebra in deg-lex orderings of generators. From Composition-Diamond lemma for associative algebras it follows that the set of Young tableaux is the Knuth normal form for plactic monoid ([30], see also Ch. 5 in [32]).

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1. Introduction

Plactic monoid is supposed to be one of the most important monoids in algebra.[2] It was introduced by D. Knuth [30] under the name "tableau" monoid ("tableau algebra") and based on Robinson 1938 and Schensted 1961 algorithm. Knuth proved that Young tableaux are normal forms, called (Robinson–Schensted–)Knuth normal forms, of elements of plactic monoids. The name "plactic" was given by Schützenberger and the basic theory of plactic monoids were given in [33]. Plactic monoids are closely connected to the representations of linear groups (Littlewood–Richardson rule), the symmetric functions (Shur functions), the quantum groups (Kashiwara crystal bases), the statistics (the charge statistics), the root systems and some others mathematical subjects.

Gröbner bases and Gröbner–Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [37,20], free Lie algebras [36,20] and implicitly free associative algebras [36,20] (see also [2,4]), by H. Hironaka [28] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [21] for ideals of the polynomial algebras.

Gröbner bases and Gröbner–Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra, see, for example, the books [1,19,22,23,26,27], the papers [2–4,7,8,10–14,21,25,29,34], and the surveys [5,6,15–18].

Let $A = \{1, 2, \ldots, n\}$ with $1 < 2 < \cdots < n$. Then we call

$$Pl(A) := sgp\langle A \mid \Omega\rangle = A^*/\equiv$$

a plactic monoid on the alphabet set A, see [32], where A^* is the free monoid generated by A, $\equiv$ is the congruence of A^* generated by the Knuth relations Ω and Ω consists of

$$ikj = kij \quad (i \le j < k), \qquad jki = jik \quad (i < j \le k).$$

Let F be a field. Then $F\langle A \mid \Omega\rangle$ is called the plactic monoid algebra over F of $Pl(A)$.

A nondecreasing word $R \in A^*$ is called a row and a strictly decreasing word $C \in A^*$ is called a column, for example, 1135556 is a row and 6531 is a column.

In the paper Okninski et al. [31] it is proved that a Gröbner–Shirshov basis of plactic monoid in initial alphabet is infinite providing the number of letters at least 4, but an explicit description of such a Gröbner–Shirshov basis is unknown.

In the paper of A.J. Cain, R. Gray and A. Malheiro [24], authors use the Schensted–Knuth normal form (the set of (semistandard) Young tableaux) to prove that the

[2] Schützenberger, Marcel-Paul, A vote for the plactic monoid. (Pour le monoïde plaxique.) (French) [J] Math. Inf. Sci. Hum. 140, 5–10 (1997). From the paper: This text is a brief answer to a question raised long ago by A. Lentin and more recently by G.-C. Rota: Why the plactic monoid ought to be considered as one of the fundamental monoids of algebra?