Nonstandard braid relations and Chebyshev polynomials

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ABSTRACT

A fundamental open problem in algebraic combinatorics is to find a positive combinatorial formula for Kronecker coefficients, which are multiplicities of the decomposition of the tensor product of two $\mathcal{S}_r$-irreducibles into irreducibles. Mulmuley and Sohoni attempt to solve this problem using canonical basis theory, by first constructing a nonstandard Hecke algebra B_r, which, though not a Hopf algebra, is a u-analogue of the Hopf algebra $\mathbb{C}\mathcal{S}_r$ in some sense (where u is the Hecke algebra parameter). For $r = 3$, we study this Hopf-like structure in detail. We define a nonstandard Hecke algebra $\check{\mathscr{H}}_3^{(k)} \subseteq \mathscr{H}_3^{\otimes k}$, determine its irreducible representations over $\mathbb{Q}(u)$, and show that it has a presentation with a nonstandard braid relation that involves Chebyshev polynomials evaluated at $\frac{1}{u+u^{-1}}$. We generalize this to Hecke algebras of dihedral groups. We go on to show that these nonstandard Hecke algebras have bases similar to the Kazhdan–Lusztig basis of $\mathscr{H}_3$ and are cellular algebras in the sense of Graham and Lehrer.

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1. Introduction

Let $\mathcal{S}_r$ denote the symmetric group on r letters and let M_ν be the $\mathcal{S}_r$-irreducible corresponding to the partition ν. The Kronecker coefficient $g_{\lambda\mu\nu}$ is the multiplicity of M_ν in the tensor product $M_\lambda \otimes M_\mu$. A fundamental and difficult open problem in algebraic combinatorics is to find a positive combinatorial formula for these coefficients. Although this problem has been studied since the early twentieth century, the general case still seems out of reach. In the last ten years this problem has seen a resurgence of effort, perhaps because of its recently discovered connections to quantum information theory [9] and complexity theory [16]. Much of the recent progress has been for Kronecker coefficients indexed by two two-row shapes, i.e., when λ and μ have two rows: an explicit, though not positive, formula was given by Remmel and Whitehead in [17] and further improvements were made by Rosas [19] and Briand, Orellana, and Rosas [7]. Briand, Orellana, and Rosas [7,8] and Ballantine and Orellana [2] have also made progress on the special case of reduced Kronecker coefficients, sometimes called the stable limit, in which the first part of the partitions λ, μ, ν is large.

In a series of recent papers, Mulmuley, in part with Sohoni and Narayanan, describes an approach to P vs. NP and related lower bound problems in complexity theory using algebraic geometry and representation theory, termed geometric complexity theory. Understanding Kronecker coefficients, particularly, having a good rule for when they are zero, is critical to their plan. In fact, Mulmuley gives a substantial informal argument claiming that if certain difficult separation conjectures like $P \neq NP$ are true, then there is a $\#P$ formula for Kronecker coefficients and a polynomial time algorithm that determines whether a Kronecker coefficient is nonzero [15]. Thus from the complexity-theoretic perspective, there is hope that Kronecker coefficients will have nice formulae like those for Littlewood–Richardson coefficients, though experience suggests they will be much harder.

A useful perspective for studying tensor products of $\mathcal{S}_r$-modules is to endow the group algebra $\mathbb{Z}\mathcal{S}_r$ with the structure of a Hopf algebra. The coproduct is $\Delta : \mathbb{Z}\mathcal{S}_r \to \mathbb{Z}\mathcal{S}_r \otimes \mathbb{Z}\mathcal{S}_r$, $w \mapsto w \otimes w$, and the $\mathbb{Z}\mathcal{S}_r$-module $M_\lambda \otimes M_\mu$ is then defined to be the restriction of the $\mathbb{Z}\mathcal{S}_r \otimes \mathbb{Z}\mathcal{S}_r$-module $M_\lambda \boxtimes M_\mu$ along Δ.

In [16], Mulmuley and Sohoni attempt to use canonical bases to understand Kronecker coefficients by constructing an algebra defined over $\mathbb{Z}[u, u^{-1}]$ that carries some of the information of the Hopf algebra $\mathbb{Z}\mathcal{S}_r$ and specializes to it at $u = 1$. Specifically, they construct the nonstandard Hecke algebra $\check{\mathscr{H}}_r$ (denoted B_r in [16]), which is a subalgebra of the tensor square of the Hecke algebra $\mathscr{H}_r$ such that the inclusion $\check{\Delta} : \check{\mathscr{H}}_r \hookrightarrow \mathscr{H}_r \otimes \mathscr{H}_r$ is a u-analogue of the coproduct Δ of $\mathbb{Z}\mathcal{S}_r$ (see Definition 2.2). The goal is then to break up the Kronecker problem into two steps [14]:

(1) Determine the multiplicity $n_{\lambda,\mu}^\alpha$ of an irreducible $\check{\mathscr{H}}_r$-module $\check{M}_\alpha$ in the tensor product $M_\lambda \otimes M_\mu$.
(2) Determine the multiplicity m_α^ν of the $\mathcal{S}_r$-irreducible M_ν in $\check{M}_\alpha|_{u=1}$.