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Taft algebras acting on associative algebras

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ABSTRACT

We examine actions of the n^2-dimensional Taft algebra H on associative algebras R. We first determine when various H-stable subrings of R contain nonzero invariants. Then we look at sufficient and necessary conditions for smash product $R\#H$ to be either semiprime or prime. This enables us to prove that if H acts on a field K such that the skew derivation is nonzero, then $K\#H$ must be a direct sum of n copies of the $n \times n$ matrices over the invariant subfield K^H.

1. Introduction

There are many results on invariants and smash products arising from the actions of group algebras, enveloping algebras, and duals of group algebras on associative algebras. All of these are examples of actions of Hopf algebras that are either commutative or cocommutative. However, less is known when the Hopf algebras are neither commutative nor commutative. Among the first and most important class of Hopf algebras of this type are the Taft algebras, which were introduced by Taft in [5]. The goal of this paper is to

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analyze the invariants and smash products obtained when Taft algebras act on associative algebras.

Section 2 examines invariants and Theorem 1 shows that, under certain conditions, H-stable subrings always have nonzero invariants. Example 2 then shows that the hypotheses in Theorem 1 are necessary. Section 3 looks at smash products $R\#H$ and Theorem 4 and Corollary 5 provide necessary and sufficient conditions for $R\#H$ to be semiprimeness. It turns out that the semiprime of $R\#H$ is determined solely by the action of a single skew derivation. Theorem 7 concludes Section 3 and it describes the structure of smash products when a Taft algebra acts on a field K.

Theorem 7. *Let K be a field acted on by the n^2-dimensional Taft algebra H such that $\delta \neq 0$. Then*

(1) $[K : K^H] = n$ *and* $K^H = K^\delta = K^\sigma = \delta^{n-1}(K)$.
(2) $K\#H$ *is isomorphic to a direct sum of n copies of the $n \times n$ matrices over* K^H.

Since Theorem 7 shows that $K\#H$ is never prime, it is natural to wonder if there are conditions that will guarantee $R\#H$ being prime. This is examined in Section 4 and Theorem 8, Corollary 9, and Corollary 10 provide necessary and sufficient conditions for $R\#H$ to be prime. Using the results from Section 4, we conclude this paper with examples where $R\#H$ is prime.

We will now introduce the notation and terminology that will be used throughout this paper. All algebras will be over a ground field k and $q \in k$ will be a primitive nth root of 1. Therefore n cannot be a multiple of the characteristic of k. The n^2-dimensional Taft algebra H is the k-algebra generated by x, y subject to the three relations $x^n = 0$, $y^n = 1$, $xy = qyx$. Thus the set $\{y^i x^j \mid 0 \leq i, j \leq n-1\}$ is a basis for H over k.

The comultiplication Δ and counit ϵ are defined on x and y as

$$\Delta(x) = x \otimes 1 + y \otimes x, \qquad \Delta(y) = y \otimes y, \qquad \epsilon(x) = 0, \qquad \epsilon(y) = 1,$$

and they can be extended uniquely to algebra homomorphisms of H. If S is the antipode of H, then

$$S(x) = -y^{-1}x, \qquad S(y) = y^{-1},$$

and S also has the interesting property that it has order $2n$.

If R is an associative algebra, then an action of the n^2-dimensional Taft algebra H on R corresponds to an automorphism σ and a q-skew σ-derivation δ satisfying the relations of the Taft algebra. This means that

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s),$$