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Almost simple groups with socle PSL(2, q) acting on abstract regular polytopes

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A B S T R A C T

Our paper deals with the classification of abstract regular polytopes for almost simple groups with socle PSL(2, q). We consider all almost simple groups PSL(2, q) $\leq G \leq$ PΓL(2, q) and determine the maximal rank of string C-group representations for G, i.e. the maximal rank of an abstract regular polytope with automorphism group G, as well as the existence of string C-group representations of lower ranks. Similar results have already been obtained by various authors in the cases $G \cong$ PSL(2, q) and $G \cong$ PGL(2, q) and they are summarized in this paper.

1. Introduction

The notion of abstract polytope generalizes the usual idea of convex polytope in Euclidean spaces. Classical highly symmetric examples include the Platonic solids, the 120-cell and the 600-cell as well as maps on surfaces. The regular (or flag-transitive)

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abstract polytopes are of particular interest. Abstract polytopes were introduced in the late 1970s under the name 'incidence polytopes'. These are particular instances of incidence complexes, a broader class of incidence structures that was introduced by Danzer and was inspired by Grünbaum's notion of polystromata. Later, Schulte laid the foundations of a general theory and initiated the focus on incidence polytopes, this being the class of incidence complexes particularly close to traditional polytopes. The general theory of regular abstract polytopes is described in a comprehensive work of McMullen and Schulte [16]. Abstract polytopes can also be seen as thin residually connected geometries in the sense of Buekenhout and Cohen [5].

Our paper deals with abstract regular polytopes whose automorphism groups are almost simple groups with socle $\mathrm{PSL}(2,q)$, the *socle* of a group G being the subgroup of G generated by its minimal normal subgroups. It is obvious that the socle of G is $\mathrm{PSL}(2,q)$ for all $\mathrm{PSL}(2,q) \leq G \leq \mathrm{P\Gamma L}(2,q)$. We thus extend the results of Leemans and Schulte [12,13]. We confirm Conjecture 4.9 in [13] and give an even more general answer in our Theorem 1.

Theorem 1. *Let* $\mathrm{PSL}(2,q) \leq G \leq \mathrm{P\Gamma L}(2,q)$ *be an almost simple group acting regularly on abstract polytopes. Then*

(1) *if* $q = 2$ *then* $G \cong \mathrm{PSL}(2,2) \cong \mathrm{S}_3$ *and* G *has a unique rank* 2 *abstract regular polytope, namely the triangle;*
(2) *if* $q = 3$ *then* $G \cong \mathrm{PGL}(2,3) \cong \mathrm{S}_4$ *and* G *acts on polytopes of rank* 3 *only;*
(3) *if* $q = 4$ *or* 5 *then either* $G \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong \mathrm{A}_5$ *and* G *acts on polytopes of rank* 3 *only, or* $G \cong \mathrm{PGL}(2,5) \cong \mathrm{S}_5$ *and* G *acts on polytopes of ranks* 3 *and* 4*;*
(4) *if* $q = 7$ *then* $G \cong \mathrm{PGL}(2,7)$ *and* G *acts on polytopes of rank* 3 *only;*
(5) *if* $q \geq 8$ *then*
   (a) *if* $q = 2^{2k+1}$, $k \geq 1$, *then* $G \cong \mathrm{PSL}(2,2^{2k+1})$ *and* G *acts on polytopes of rank* 3 *only;*
   (b) *if* $q = 9$ *then either* $G \cong \mathrm{PGL}(2,9)$, *or* $G \cong \mathrm{P\Sigma L}(2,9) \cong \mathrm{S}_6$, *or* $G \cong \mathrm{P\Gamma L}(2,9)$, *and* G *acts on polytopes of rank* 3*; moreover* $\mathrm{P\Sigma L}(2,9)$ *acts on polytopes of ranks* 3, 4 *and* 5*;*
   (c) *if* $q = p^{2k+1} \geq 11$, p *an odd prime and* $k \geq 0$, *then* $G \cong \mathrm{PSL}(2,p^{2k+1})$ *or* $G \cong \mathrm{PGL}(2,p^{2k+1})$*; in either case,* G *acts on polytopes of rank* 3*; if moreover* $q = 11$ *or* 19 *then* $G \cong \mathrm{PSL}(2,q)$ *acts on polytopes of rank* 4*;*
   (d) *if* $q = p^{2k} \geq 16$, p *any prime and* $k \geq 1$ *then either* $G \cong \mathrm{PSL}(2,p^{2k})$ *or* $G \cong \mathrm{PGL}(2,p^{2k})$ *or* $G \cong \mathrm{PSL}(2,p^{2k}) \rtimes \langle\beta\rangle$ *or* $G \cong \mathrm{PGL}(2,p^{2k}) \rtimes \langle\beta\rangle$, *where* β *is a Baer involution of* $\mathrm{P\Gamma L}(2,p^{2k})$*; in all four cases,* G *acts on polytopes of rank* 3*; moreover,* $\mathrm{PSL}(2,p^{2k}) \rtimes \langle\beta\rangle$ *acts on polytopes of rank* 4.

Cases 1 to 4 are folklore in the theory of abstract regular polytopes; we state them for the sake of completeness. Cases 5(a)–(c) follow from Sjerve and Cherkassoff [18], Leemans and Vauthier [15] and Leemans and Schulte [12,13] and the observation that no overgroup