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Compatible pairs of Borel subalgebras and shared orbit pairs

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ABSTRACT

Consider a class of pairs $(\mathfrak{g}, \mathfrak{g}_1)$, where $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{g}_1$ is a subalgebra reductive in $\mathfrak{g}$, satisfying the following: For any Cartan subalgebra $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ there is a unique Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ containing $\mathfrak{h}_1$. Given such a pair $(\mathfrak{g}, \mathfrak{g}_1)$ and a Borel subalgebra $\mathfrak{b}_1 \subseteq \mathfrak{g}_1$ we study a (finite) set $\mathcal{S}^{\mathfrak{g}}_{\mathrm{Bor}}(\mathfrak{b}_1)$ of all Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$ containing $\mathfrak{b}_1$. In particular we point at the subclass of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ when $\mathcal{S}^{\mathfrak{g}}_{\mathrm{Bor}}(\mathfrak{b}_1)$ is a singleton, for every Borel subalgebra $\mathfrak{b}_1 \subseteq \mathfrak{g}_1$. As a consequence, for such pairs we relate the corresponding flag varieties $\mathcal{B}(\mathfrak{g})$ and $\mathcal{B}(\mathfrak{g}_1)$. As an interesting class of pairs $(\mathfrak{g}, \mathfrak{g}_1)$, for which we can apply our results on pairs of Borel subalgebras, we study in detail the pairs considered by R. Brylinski and B. Kostant related to shared orbit pairs.

Introduction

Unless specified otherwise throughout this paper $\mathbb{K}$ denotes a field of characteristic zero. All Lie algebras we consider are finite dimensional over the ground field. Given

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a Lie algebra $\mathfrak{g}$, we define $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \overline{\mathbb{K}}$, where $\overline{\mathbb{K}}$ is a fixed algebraic closure of $\mathbb{K}$. By $B_{\mathfrak{g}}$ we denote the Killing form of $\mathfrak{g}$. Given reductive $\mathfrak{g}$ and a split Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, by $\Delta(\mathfrak{g}, \mathfrak{h})$ we denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. By $\Pi(\mathfrak{g}, \mathfrak{h})$ we denote a choice of simple roots. For a root ϕ, X_ϕ denotes a nonzero root vector from the root subspace $\mathfrak{g}_\phi$. By $\Delta^\pm = \Delta^\pm(\mathfrak{g}, \mathfrak{h})$ we denote the positive/negative roots, and then $\mathfrak{n}^\pm = \mathfrak{n}^\pm(\mathfrak{g}) = \sum_{\phi \in \Delta^\pm} \mathfrak{g}_\phi$. Thus we have the usual triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

For a semisimple Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{g}_1$ which is reductive in $\mathfrak{g}$, it is worth to know when the pair $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies the following condition:

(**Q1**) *For any Cartan subalgebra* $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ *there exists a* unique *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ *containing* $\mathfrak{h}_1$.

The main goal of the present paper is to study those pairs satisfying the following stronger condition:

(**Bor**) *The pair* $(\mathfrak{g}, \mathfrak{g}_1)$ *satisfies* (**Q1**)*, and for any Borel subalgebra* $\mathfrak{b}_1 \subseteq \mathfrak{g}_1$ *there exists a* unique *Borel subalgebra* $\mathfrak{b} \subseteq \mathfrak{g}$ *containing* $\mathfrak{b}_1$.

For a pair of Cartan subalgebras $(\mathfrak{h}, \mathfrak{h}_1)$ as in (**Q1**), and a chosen Borel subalgebra $\mathfrak{b}_1$ of $\mathfrak{g}_1$ containing $\mathfrak{h}_1$, we thus have a unique Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$ and $\mathfrak{b}_1$. We say that such a pair of Borel subalgebras $(\mathfrak{b}, \mathfrak{b}_1)$ is *compatible* with the pair of Cartan subalgebras $(\mathfrak{h}, \mathfrak{h}_1)$; or just that it is compatible.

Suppose again that $\mathfrak{g}$ is semisimple and $\mathfrak{g}_1 \subseteq \mathfrak{g}$ is reductive in it. For simplicity we will here also assume that $\mathbb{K}$ is algebraically closed. It is interesting to know whether $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies the condition (**Bor**). And if not, given a Borel subalgebra $\mathfrak{b}_1$ of $\mathfrak{g}_1$, can we determine the set of all Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{b}_1$? Let us be more precise. For any reductive $\mathbb{K}$-Lie algebra $\mathfrak{r}$ by $\mathcal{B}(\mathfrak{r})$ we denote the set of all its Borel subalgebras. Now assume that a pair $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies the condition (**Bor**). Then the map

$$\mathsf{B} = \mathsf{B}^{\mathfrak{g}}_{\mathfrak{g}_1} : \mathcal{B}(\mathfrak{g}_1) \to \mathcal{B}(\mathfrak{g}),$$

given by $\mathsf{B}(\mathfrak{b}_1) = \mathfrak{b}$, with $\mathfrak{b}_1$ and $\mathfrak{b}$ as in (**Bor**), is well defined. More generally now suppose that a pair $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies (**Q1**). For any $\mathfrak{b}_1 \in \mathcal{B}(\mathfrak{g}_1)$ by $\widehat{\mathsf{B}}(\mathfrak{b}_1)$, or $\mathcal{S}^{\mathfrak{g}}_{\mathrm{Bor}}(\mathfrak{b}_1)$, we denote the subset of $\mathcal{B}(\mathfrak{g})$ consisting of all Borel subalgebras $\mathfrak{b}$ containing $\mathfrak{b}_1$; i.e., consider a map

$$\widehat{\mathsf{B}} = \widehat{\mathsf{B}}^{\mathfrak{g}}_{\mathfrak{g}_1} : \mathfrak{b}_1 \mapsto \widehat{\mathsf{B}}(\mathfrak{b}_1).$$

The following theorem is our first main result. It summarizes our knowledge about the class of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ satisfying (**Bor**), and the above defined maps B and $\widehat{\mathsf{B}}$; see Corollary 2.10 and Propositions 2.11, 2.15 and 2.17. In particular the part (iii) is a geometric statement saying that for a pair $(\mathfrak{g}, \mathfrak{g}_1)$ satisfying (**Bor**) the flag variety $\mathcal{B}(\mathfrak{g}_1)$ sits inside