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Regular subalgebras and nilpotent orbits of real graded Lie algebras

Heiko Dietrich[a,*,1], Paolo Faccin[b], Willem A. de Graaf[b]

[a] School of Mathematical Sciences, Monash University, VIC 3800, Australia
[b] Department of Mathematics, University of Trento, Povo (Trento), Italy

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ABSTRACT

For a semisimple Lie algebra over the complex numbers, Dynkin (1952) developed an algorithm to classify the regular semisimple subalgebras, up to conjugacy by the inner automorphism group. For a graded semisimple Lie algebra over the complex numbers, Vinberg (1979) showed that a classification of a certain type of regular subalgebras (called carrier algebras) yields a classification of the nilpotent orbits in a homogeneous component of that Lie algebra. Here we consider these problems for (graded) semisimple Lie algebras over the real numbers. First, we describe an algorithm to classify the regular semisimple subalgebras of a real semisimple Lie algebra. This also yields an algorithm for listing, up to conjugacy, the carrier algebras in a real graded semisimple real algebra. We then discuss what needs to be done to obtain a classification of the nilpotent orbits from that; such classifications have applications in differential geometry and theoretical physics. Our algorithms are implemented in the language of the computer algebra system GAP, using our package CoReLG; we report on example computations.

* Corresponding author.
E-mail addresses: heiko.dietrich@monash.edu (H. Dietrich), faccin@science.unitn.it (P. Faccin), degraaf@science.unitn.it (W.A. de Graaf).

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1. Introduction

Let $\mathfrak{g}^c$ be a complex semisimple Lie algebra with adjoint group G^c. Classifying the semisimple subalgebras of $\mathfrak{g}^c$ up to G^c-conjugacy is an extensively studied problem, partly motivated by applications in theoretical physics; see for example [12,19,20,41,42]. In [20,41], this classification is split into two parts: the construction of regular semisimple subalgebras, that is, semisimple subalgebras normalised by a Cartan subalgebra of $\mathfrak{g}^c$, and the construction of semisimple subalgebras not contained in any regular proper subalgebra. Dynkin [20] presented, among other things, an algorithm to list the regular semisimple subalgebras of $\mathfrak{g}^c$, up to G^c-conjugacy. One of the main facts underpinning this algorithm is that two semisimple subalgebras, normalised by the same Cartan subalgebra $\mathfrak{h}^c$, are G^c-conjugate if and only if their root systems are conjugate under the Weyl group of the root system of $\mathfrak{g}^c$ (with respect to $\mathfrak{h}^c$). The situation is more intricate for a real semisimple Lie algebra $\mathfrak{g}$. As a consequence, here the aim is usually not to classify the semisimple subalgebras, but to decide whether a given real form $\mathfrak{a}$ of a complex subalgebra $\mathfrak{a}^c$ of $\mathfrak{g}^c = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is contained in $\mathfrak{g}$, see [10,11,21,22,24,37]. However, for some classes of subalgebras a classification up to G-conjugacy can be obtained, with G the adjoint group of $\mathfrak{g}$. Examples are the subalgebras isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, whose classification up to G-conjugacy is equivalent to classifying the nilpotent orbits in $\mathfrak{g}$; the latter can be performed using the Kostant–Sekiguchi correspondence, see [9,16]. It is the first aim of this paper to show that also the regular semisimple subalgebras of $\mathfrak{g}$ can be classified up to G-conjugacy; we describe an effective algorithm for this task. The main issue is that, in general, there exist Cartan subalgebras of $\mathfrak{g}$ which are not G-conjugate, and that a given regular semisimple subalgebra can be normalised by several non-conjugate Cartan subalgebras. To get some order in this situation, we introduce the notion of "strong $\mathfrak{h}$-regularity"; we show that two strongly $\mathfrak{h}$-regular subalgebras are G-conjugate if and only if their root systems are conjugate under the real Weyl group of $\mathfrak{g}$ (with respect to $\mathfrak{h}$).

The second problem motivating this paper is the determination of the nilpotent orbits in a homogeneous component of a graded semisimple Lie algebra. Over the complex numbers (or, more generally, over an algebraically closed field of characteristic 0), the theory of orbits in a graded semisimple Lie algebra has been developed by Vinberg [49,50,48]. Let $\mathfrak{g}^c = \bigoplus_{i\in\mathbb{Z}_m} \mathfrak{g}^c_i$ be a graded semisimple complex Lie algebra, where we write $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, and $\mathbb{Z}_m = \mathbb{Z}$ when $m = \infty$. The component $\mathfrak{g}^c_0$ is a reductive subalgebra of $\mathfrak{g}^c$, and G^c_0 is defined as the connected subgroup of G^c with Lie algebra $\mathfrak{g}^c_0$. This group acts on the homogeneous component $\mathfrak{g}^c_1$, and the question is what its orbits are. It turns out that many constructions regarding the action of G^c on $\mathfrak{g}^c$ can be generalised to this setting. In particular, there exists a Jordan decomposition, so that the orbits of G^c_0 in $\mathfrak{g}^c_1$ naturally split into three types: nilpotent, semisimple, and mixed. Of special interest are nilpotent orbits: in contrast to semisimple orbits, there exist only finitely many of them. It is known that every nonzero nilpotent $e \in \mathfrak{g}^c_1$ lies in a homogeneous $\mathfrak{sl}_2$-triple (h, e, f), with $h \in \mathfrak{g}^c_0$ and $f \in \mathfrak{g}^c_{-1}$, and that G^c_0-conjugacy of