Journal of Algebra 423 (2015) 1080–1108

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Algorithmic recognition of quasipositive braids of algebraic length two

S.Yu. Orevkov [a,b,*]

[a] IMT, Université Toulouse-3, France
[b] Steklov Math. Institute, Moscow, Russia

ARTICLE INFO

Article history:
Received 3 June 2014
Available online 23 October 2014
Communicated by Patrick Dehornoy

Keywords:
Braid group
Garside group
Quasipositive braid

ABSTRACT

We give an algorithm to decide if a given braid is a product of two factors which are conjugates of given powers of standard generators of the braid group. The same problem is solved in a certain class of Garside groups including Artin–Tits groups of spherical type. The solution is based on the Garside theory and, especially, on the theory of cyclic sliding developed by Gebhardt and González-Meneses. We show that if a braid is of the required form, then any cycling orbit in its sliding circuit set in the dual Garside structure contains an element for which this fact is immediately seen from the left normal form.

Introduction

Let Br_n be the braid group with n strings. It is generated by $\sigma_1, \dots, \sigma_{n-1}$ (called *standard* or *Artin* generators) subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1; \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{for } |i-j| = 1.$$

In this paper we give an algorithm (rather efficient in practice) to decide if a given braid is the product of two factors which are conjugates of given powers of standard generators.

* Correspondence to: IMT, Université Toulouse-3, France.
E-mail address: orevkov@math.ups-tlse.fr.

http://dx.doi.org/10.1016/j.jalgebra.2014.10.004
0021-8693/

Since our solution is based on Garside theory, as a by-product we obtain a solution to a similar problem for a certain class of Garside groups which includes Artin–Tits groups of spherical type (we call them in this paper just Artin groups; note that Br_n is the Artin group of type A_{n-1}). The main ingredient of our solution is the theory of cyclic sliding developed by Gebhardt and González-Meneses in [17]. In fact, we show that if an element X is a product of two conjugates of atom powers, then its set of sliding circuits $\mathrm{SC}(X)$ contains an element for which this property is immediately seen from the left normal form. If the Garside structure is symmetric (which is the case for the dual structures on Artin groups), then any cycling orbit in $\mathrm{SC}(X)$ contains such an element.

When speaking of Garside groups, we use mostly the terminology and notation from [17]. All necessary definitions and facts from the Garside theory are given in Section 1 below. For readers familiar with the Garside theory, we just say here that by a Garside structure on a group G we mean a triple $(G, \mathcal{P}, \Delta)$ were $\mathcal{P}$ is the submonoid of positive elements and Δ is the Garside element (see details in Section 1). The letter length function on $\mathcal{P}$ is denoted by $\|\ \|$ and the set of atoms is denoted by $\mathcal{A}$.

It is convenient also to give the following new definitions. We say that a Garside structure is *symmetric* if, for any two simple elements u, v, one has $(u \prec v) \Leftrightarrow (v \succ u)$. The main example is the dual Garside structure on Artin groups introduced by Bessis [1], see [1, §1.2]. In particular, the Birman–Ko–Lee Garside structure on the braid groups [4] is symmetric. Another example is the braid extension of the complex reflection group $G(e, e, r)$ with the Garside structure introduced in [2].

Following [6], we say that $X \in \mathcal{P}$ is *square free* if there do not exist $U, V \in \mathcal{P}$ and $x \in \mathcal{A}$ such that $X = Ux^2V$. A Garside structure is called *square free* if all simple elements are square free. We say that a Garside structure is *homogeneous* if $\|XY\| = \|X\| + \|Y\|$ for any $X, Y \in \mathcal{P}$, thus, $\|\ \|$ extends up to a unique homomorphism $e : G \to \mathbb{Z}$ such that $e|_{\mathcal{A}} = 1$. Both the standard and the dual Garside structure on Artin groups are square free and homogeneous.

The conjugacy class of an element X of a group G is denoted by X^G. We use Convention 1.8 (see Section 1 below) for the presentation of left (right) normal forms. Let us give the statements of the main results (the proofs are in Sections 3 and 4).

Theorem 1. *Let $(G, \mathcal{P}, \delta)$ be a symmetric homogeneous Garside structure of finite type with set of atoms $\mathcal{A}$. Let k, l be positive integers. When $k \geq 2$ in part* (a) *or when* $\max(k, l) \geq 2$ *in part* (b)*, we suppose in addition that the Garside structure is square free. Let $X \in G$ and $x, y \in \mathcal{A}$. Then:*

(a) *$X \in (x^k)^G$ if and only if the left normal form of X is*

$$\delta^{-n} \cdot A_n \cdot \ldots \cdot A_2 \cdot A_1 \cdot x_1^k \cdot B_1 \cdot B_2 \cdot \ldots \cdot B_n \tag{1}$$

where $n \geq 0$, $x_1 \in x^G \cap \mathcal{A}$ and A_i, B_i are simple elements such that

$$A_i \delta^{i-1} B_i = \delta^i, \quad i = 1, \ldots, n. \tag{2}$$