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Bounded regularity

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A B S T R A C T

Let k be a field and S the polynomial ring $k[x_1, \dots, x_n]$. For a non-trivial finitely generated homogeneous S-module M with grading in $\mathbb{Z}$, an integer D and some homogeneous polynomial f in S, it is defined what it means that f is *regular on M up to degree D*. Following the usual definition of regularity, a generalization to finite sequences of polynomials in S is given.

Different criteria for a finite sequence of polynomials in S to be regular up to a particular degree are given: first a characterization with Hilbert series, then a characterization with first syzygies, and finally, for $M = S$, characterizations with Betti numbers as well as with the Koszul complex and free resolutions.

1. The notion of bounded regularity

Let k be a field and $S = k[x_1, \dots, x_n]$ be a polynomial ring over k. We consider finite sequences of homogeneous polynomials of positive degrees and their operation on non-trivial finitely generated graded S-modules (with grading in $\mathbb{Z}$). Here and in the

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following, by a polynomial, we always mean an element of S. Moreover, by an S-module we mean in the following a graded S-module with grading in $\mathbb{Z}$.

If now M is such a module and additionally M has finite length, that is, $M_n = 0$ for $n \gg 0$, then clearly, there is no regular element on M. It might however be the case that a homogeneous polynomial is regular on M "up to a particular degree". This motivates our basic definition:

Definition 1. Let M be a non-trivial finitely generated S-module.

Let f be a homogeneous polynomial of degree d and D an integer. Then f is *regular up to degree D on M* if f is non-constant and for all $i \leq D - d$, the linear map $M_i \longrightarrow M_{i+d}$ given by multiplication with f is injective.

More generally, let $f_1, \ldots, f_r$ be a sequence of homogeneous polynomials and $D \in \mathbb{Z}$. Then the sequence is *regular up to degree D* on M if all the polynomials are non-constant and for each $q = 1, \ldots, r$, f_q is regular on $M/(f_1, \ldots, f_{q-1})M$ up to degree D.

Following the usual terminology, a sequence is simply called *regular up to degree D* if it fulfills the definition with $S = M$.

Let M and $f_1, \ldots, f_r$ be as in the definition. As M is assumed to be finitely generated, $d_{\min} := \min\{i \in \mathbb{Z} \mid M_i \neq 0\}$ exists. As the polynomials are homogeneous of positive degree, we then have $(M/(f_1, \ldots, f_r)M)_{d_{\min}} = M_{d_{\min}} \neq 0$. Therefore, $M/(f_1, \ldots, f_r)M \neq 0$ which is a necessary condition in order that a sequence be regular; cf. [11,6]. Clearly, the system is regular if and only if it is regular up to degree D for each $D \in \mathbb{Z}$. This confirms that our definition is reasonable.

Let us call a Laurent series over $\mathbb{Z}$ in the variable t simply a Laurent series, and let us extend the meaning of $a \equiv b \bmod t^d$ for two Laurent series a, b and $d \in \mathbb{N}$ to any $d \in \mathbb{Z}$ in the obvious way: $a \equiv b \bmod t^d$ if and only if the t-valuation of $a - b$ is at least d.

Let now M be as above and let f be a homogeneous polynomial of positive degree d. Furthermore, let $H_M = H_M(t) = \sum_{i=-\infty}^{\infty} \dim_k(M_i)t^i$ be the Hilbert series of M, which is a Laurent series over $\mathbb{Z}$ defined by a rational function.

It is immediate that f is regular up to degree D on M if and only if $H_{M/fM} \equiv (1-t^d) \cdot H_M \bmod t^{D+1}$. Therefore, if the sequence $f_1, \ldots, f_r$ with homogeneous polynomials of degrees $d_1, \ldots, d_r$ is regular up to degree D on M then $H_{M/(f_1,\ldots,f_r)M} \equiv \prod_{i=1}^{r}(1 - t^{d_i}) \cdot H_M \bmod t^{D+1}$. We will prove, in Section 4, that the converse of this statement also holds. This establishes in particular that regularity up to a particular degree is independent of the ordering of the polynomials.

Further contributions in this article are: In the fifth section, we give characterizations of regularity up to some degree in terms of first syzygies. In the sixth and last section, we characterize regularity up to some degree on S itself in terms of Betti numbers as well as in terms of the Koszul complex and free resolutions. We also prove some general results on complexes, in particular on the Koszul complex, from a "bounded degree" point of view.